AVERAGE POWER AND EFFECTIVE VALUES

Average Power : For a periodic function with period the average value is given as

$$P = \frac{1}{T}\int_{t_x}^{t_x+T} p(t)\,dt$$

or

$$P = \frac{1}{nT}\int_{t_x}^{t_x+nT} p(t)\,dt = \frac{1}{nT}\int_{-nT/2}^{nT/2} p(t).dt = \lim_{n\to\infty}\frac{1}{nT}\int_{-\frac{nT}{2}}^{\frac{nT}{2}} p(t)dt$$

Replacing nT by the continuous variable τ

$$P = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} p(t)\,dt$$

Now, let's look at general case. We can write

$$v(t) = V_m\cos(\omega t+\alpha)$$

$$i(t) = I_m\cos(\omega t+\beta) = I_m\cos(\omega t+\alpha-\theta)$$

for any element. Note that $\beta = \alpha-\theta \rightarrow \theta = \alpha-\beta$ is the phase angle by which the voltage leads the current.

$$p(t) = v(t).i(t) = V_m I_m\cos(\omega t+\alpha)\cos(\omega t+\alpha-\theta)$$

Using the identity $\cos a.\cos b = \frac{1}{2}[\cos(a-b)+\cos(a+$

$$p(t) = \frac{1}{2}I_m V_m\left[\cos\theta + \cos(2\omega t+2\alpha-\theta)\right]$$

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m\left[\cos\theta + \cos(2\omega t+2\alpha-\theta)\right]dt$$

$$= \frac{V_m I_m}{2}\cdot\underbrace{\frac{1}{T}\cos\theta\int_0^T dt}_{1} + \frac{V_m I_m}{2T}\underbrace{\int_0^T \cos(2\omega t+2\alpha-\theta)\,dt}_{0}$$

$$\boxed{P = \frac{1}{2}V_m I_m\cos\theta}$$

I-2

Case 1: For an ideal resistor, $v = Ri$, and $\theta = 0$:

$$P_R = \frac{1}{2} V_m I_m \cos 0^\circ = \frac{1}{2} V_m I_m = \frac{1}{2} R I_m^2 = \frac{1}{2} \frac{V_m^2}{R}$$

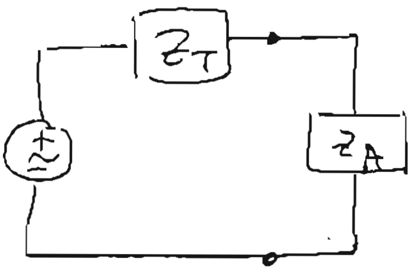
Case 2: For an ideal reactive element $\theta = \mp 90^\circ$,

$$P_X = 0$$

Ex: Let $Z = 5 + j3$ and $\vec{I} = 2\angle 25^\circ$ A, then

$$P = \frac{1}{2} \cdot 5 \cdot 2^2 = 10 \text{ W}$$

Maximum Power Transfer:



$$Z_T = R_T + jX_T$$
$$Z_A = R_A + jX_A$$

For maximum power transfer,

$X_A = -X_T$ and $R_A = R_T$, or

$$Z_A = Z_T^*$$

Power for cases where $i(t)$ is not periodic:

Let $i(t) = \cos t + \cos\sqrt{2}t$ and $R = 1\ \Omega$. Then

see "EK" ★

$$P = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \left(\cos^2 t + 2\cos t \cdot \cos\sqrt{2}t + \cos^2\sqrt{2}t\right) dt$$

$$= \underbrace{\lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \cos^2 t\, dt}_{\frac{1}{2}} + \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \cos t \cdot \cos\sqrt{2}t\, dt + \underbrace{\lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \cos^2\sqrt{2}t}_{\frac{1}{2}}$$

The middle term's integrand can be expressed as

$$\frac{1}{2}\left[\cos(1+\sqrt{2})t + \cos(1-\sqrt{2})t\right]$$

and will result in zero when integrated.

$$P = \frac{1}{2} + 0 + \frac{1}{2} = 1 \text{ W}$$

We can generalize this result.

$$i(t) = I_{m1}\cos\omega_1 t + I_{m2}\cos\omega_2 t + \cdots + I_{mn}\cos\omega_n t$$

will deliver an average power, to a resistor R, of

$$P = \frac{1}{2}\left(I_{m1}^2 + I_{m2}^2 + \cdots + I_{mn}^2\right) R$$

Note: The sinusoids should be at different frequencies to use the above formula.

Ex: $R = 5\,\Omega$, $i(t) = 4\cos 2t - 6\cos 4t$

$$P = \frac{1}{2}\left(4^2 + 6^2\right) 5 = 130 \text{ W}$$

If $i(t) = 4\cos 2t - 6\cos 2t$, then

$$i(t) = -2\cos 2t$$

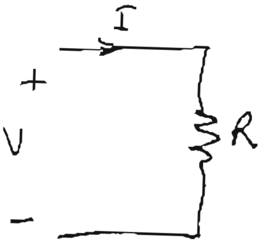
$$P = \frac{1}{2}\left(2^2\right) 5 = 10 \text{ W}$$

Note: The last function can be expressed in infinitely many different ways, such as

$$i(t) = 999998\cos 2t - 1000000\cos 2t$$

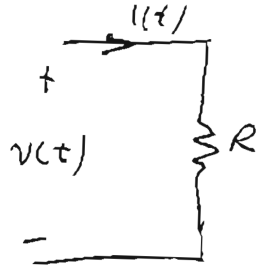
This would not mean, we can obtain a very large power $\left[P \neq \frac{1}{2}\left(999998^2 + 1000000^2\right)R\right]$

Effective values of current and voltage:



DC Case

$P = R I^2$



AC case

$$P = \frac{1}{T} \int_0^T R i^2 \, dt$$

The question is "Can we express the average power in a similar fashion to DC case?"

$$P = \frac{1}{T} \int_0^T R i^2 \, dt = R I_e^2$$ where I_e is the effective value of the current. Then

$$I_e = \sqrt{\frac{1}{T} \int_0^T i^2 \, dt}$$

Similarly

$$V_e = \sqrt{\frac{1}{T} \int_0^T v^2 \, dt}$$

* The effective values are also called rms values (root-mean-square).

Ex $\quad i(t) = I_m \cos(\omega t + \beta)$, $\quad \omega = \frac{2\pi}{T}$

$$I_e = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2(\omega t + \beta) \, dt} = \sqrt{\frac{I_m^2}{2T} \int_0^T [1 + \cos(2\omega t + 2\beta)] \, dt}$$

$$= \sqrt{\frac{I_m^2}{2T} \left[t + \frac{\sin(2\omega t + 2\beta)}{2\omega} \right]_0^T} = \sqrt{\frac{I_m^2}{2T} \left[T + \frac{\sin(2\omega T + 2\beta) - \sin 2\beta}{2\omega} \right]}$$

$$= \sqrt{\frac{I_m^2}{2T} \left[T + \frac{\sin 2\beta - \sin 2\beta}{2\omega} \right]} = \frac{I_m}{\sqrt{2}}$$

Then

$$P = \frac{1}{2} R I_m^2 = R I_e^2 = V_e I_e = \frac{V_e^2}{R}$$

The general expression for average power will become

$$P = \frac{1}{2} V_m I_m \cos\theta = V_e I_e \cos\theta$$

Ex: The voltage we use in homes is 220 V_{rms} in Turkey. What is its peak value?

$$V_m = \sqrt{2}\, V_e = \sqrt{2} \cdot 220 = 311 \text{ V}$$

* Phasor currents and voltages can also be expressed with their effective values.

$$\vec{I} = 34 \angle 20^\circ \text{ A} \rightarrow \vec{I}_e = \frac{34}{\sqrt{2}} \angle 20^\circ \text{ A}_{rms}$$

Note: ~~[illegible]~~ Currents which are the sum of sinusoidal currents with different frequencies will have an effective value of

$$I_e = \sqrt{I_{1e}^2 + I_{2e}^2 + \cdots + I_{ne}^2}$$

Apparent Power:

If $v(t) = V_m \cos(\omega t + \alpha)$ and $i(t) = V_m \cos(\omega t + \beta)$ with $\theta = \alpha - \beta$, then we know that

$$P = \frac{1}{2} V_m I_m \cos\theta = V_e I_e \cos\theta$$

The product $V_e I_e$ is called the apparent power, since it is the maximum power that can be obtained from the circuit, $\cos\theta = 1$ is the maximum value of $\cos\theta$.

That is, $V_e I_e$ is the apparent power, because the actual power that will be obtained depends on $\cos\theta$.

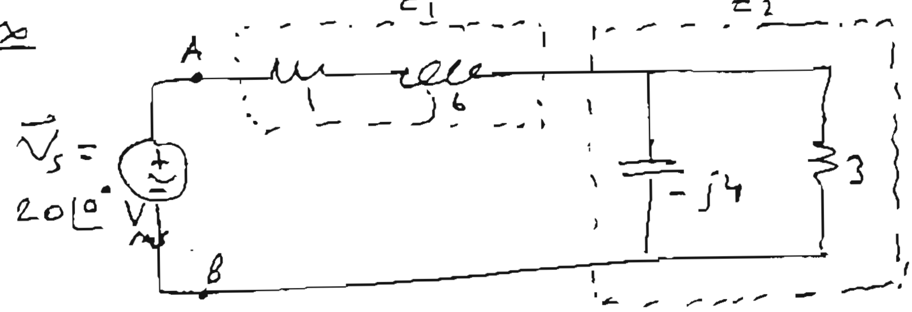
Power Factor:

$$P.F. = \frac{\text{average power}}{\text{apparent power}} = \frac{P}{V_e I_e} = \cos\theta$$

θ = power factor angle

Note: For a given PF, we would not be able to tell if θ is positive or negative. Hence load type should also be specified when giving PF. Such as PF = 0.8 inductive

Ex



$Z_1 = 1 + j6$

$$Z_2 = \frac{3(-j4)}{3-j4} = \frac{-j12}{3-j4} = \frac{-j12(3+j4)}{3^2+4^2}$$

$$= 1.92 - j1.44$$

$$Z_{AB} = 1 + j6 + 1.92 - j1.44 = 2.92 + j4.56$$

$$\vec{I} = \frac{20\angle 0^\circ}{2.92 + j4.56} = \frac{20\angle 0^\circ}{5.41\angle 57.4^\circ} = 3.70\angle -57.4^\circ$$

$$P_s = (20)(3.70)\cos[0 - (-57.4^\circ)] = 74\cos 57.4^\circ = 39.9\,W$$

$PF = \cos 57.4^\circ = 0.54$ Inductive

Note: $Z_{AB} = 5.41\angle 57.4^\circ$, hence $PF = \cos 57.4^\circ = 0.54$

since $Z_{AB} = \frac{\vec{V}_s}{\vec{I}} = \left|\frac{\vec{V}_s}{\vec{I}}\right| \angle\theta$.

Complex Power:

$$P = I_e V_e \cos\theta = I_e V_e \cos(\alpha - \beta) = I_e V_e \, RE\left[e^{j(\alpha-\beta)}\right]$$

$$= RE\left[I_e V_e \, e^{j(\alpha-\beta)}\right] = RE\left[I_e e^{-j\beta} \cdot V_e e^{j\alpha}\right]$$

$$= RE\left[\vec{I}_e^{\,*} \cdot \vec{V}_e\right]$$

S = Complex power = $\vec{V}_e \vec{I}_e^{\,*}$

$|S| = V_e I_e$ = apparent power

$\angle S = \alpha - \beta = \theta$ = P.F. angle

and written in Cartesian coordinates:

$$S = \vec{V}_e \vec{I}_e^{\,*} = V_e I_e \, e^{j(\alpha-\beta)} = V_e I_e \, e^{j\theta}.$$

$$= V_e I_e \cos\theta + j\, V_e I_e \sin\theta$$

$$= P + jQ$$

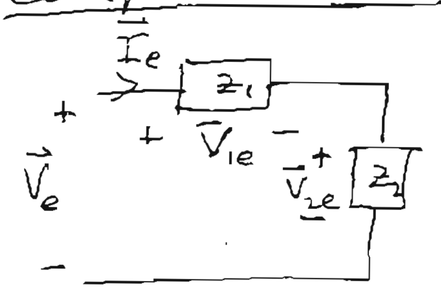
P ← average power

Q ← reactive power (VAR)

<u>Note</u>: $\vec{I}_e = I_e \cos\theta + jI_e \sin\theta = \vec{I}_{e1} + \vec{I}_{e2}$ where $\vec{I}_{e1}$ is the in phase component with $\vec{V}_e$ and $\vec{I}_{e2}$ is the $90°$ out of phase (perpendicular) component of $\vec{I}_e$. Hence

$$P = |\vec{V}_e \cdot \vec{I}_{e1}| \quad \text{and} \quad Q = |\vec{V}_e \cdot \vec{I}_{e2}|$$

<u>Complex Power in Load Combinations</u>:

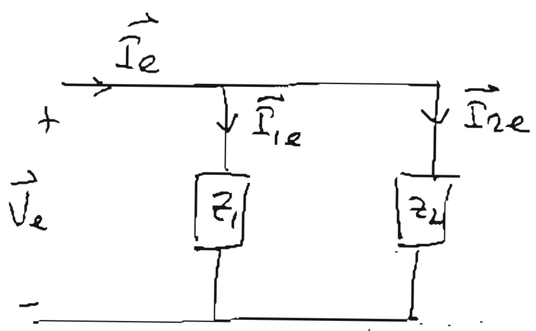


$$\vec{V}_e = \vec{V} + \vec{V}.$$

$$S = \vec{V}_e \vec{I}_e^{\,*} = (\vec{V}_{1e} + \vec{V}_{2e}) \vec{I}_e^{\,*}$$

$$= \vec{V}_{1e} \vec{I}_e^{\,*} + \vec{V}_{2e} \vec{I}_e^{\,*}$$

$$= S_1 + S_2$$



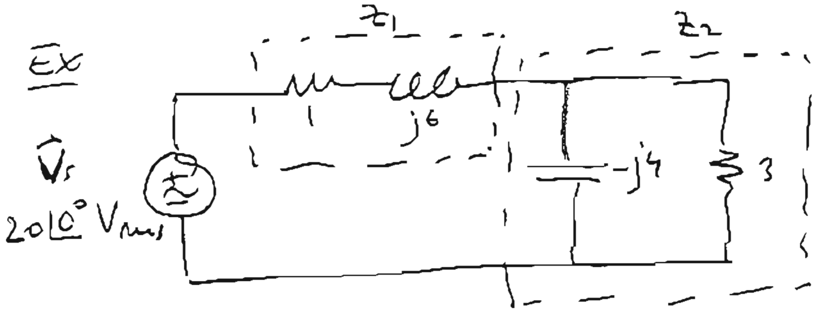
$$\vec{I}_e = \vec{I}_{1e} + \vec{I}_{2e}$$

$$S = \vec{V}_e \vec{I}_e^{\,*} = \vec{V}_e (\vec{I}_{1e} + \vec{I}_{2e})^*$$

$$= \vec{V}_e \vec{I}_{1e}^{\,*} + \vec{V}_e \vec{I}_{2e}^{\,*}$$

$$= S_1 + S_2$$

Ex



$$Z_1 = 1 + j6$$

$$Z_2 = \frac{3(-j4)}{3-j4} = \frac{-j12}{3-j4} = \frac{-j12(3+j4)}{25} = 1.92 - j1.$$

$$Z_{AB} = 1 + j6 + 1.92 - j1.44 = 2.92 + j4.56\ \Omega$$

$$\vec{I}_s = \frac{20\angle 0^\circ}{2.92 + j4.56} = \frac{20\angle 0^\circ}{5.41\angle 57.4^\circ} = 3.69\angle -57.4^\circ\ A_{rms}$$

$$S_s = 20\angle 0^\circ \cdot (3.69\angle 57.4^\circ) = 73.8\angle 57.4^\circ = 39.8 + j\,62.2$$

$$P_s = 39.8\ W, \qquad Q_s = 62.2\ VAR$$

$$S_{Z_1} = \vec{V}_{Z_1}\vec{I}_{Z_1}^{\,*} = Z_1 \vec{I}_{Z_1} \cdot \vec{I}_{Z_1}^{\,*} = Z_1 |\vec{I}_{Z_1}|^2$$

$$= (1+j6)(3.69)^2 = 13.6 + j\,81.7$$

$$S_{Z_2} = (1.92 - j1.44)(3.69)^2 = 26.1 - j\,19.6$$

Hence $P_{Z_1} = 13.6\,W$, $P_{Z_2} = 26.1\,W$, $P_{Total} = 13.6 + 26.1 = 39.7\,W$

$= P_s$

$Q_{Z_1} = 81.7\ VAR$, $Q_{Z_2} = -19.6\ VAR$

$Q_{Total} = 81.7 - 19.6 = 62.1\ VAR$

$\simeq Q_s$

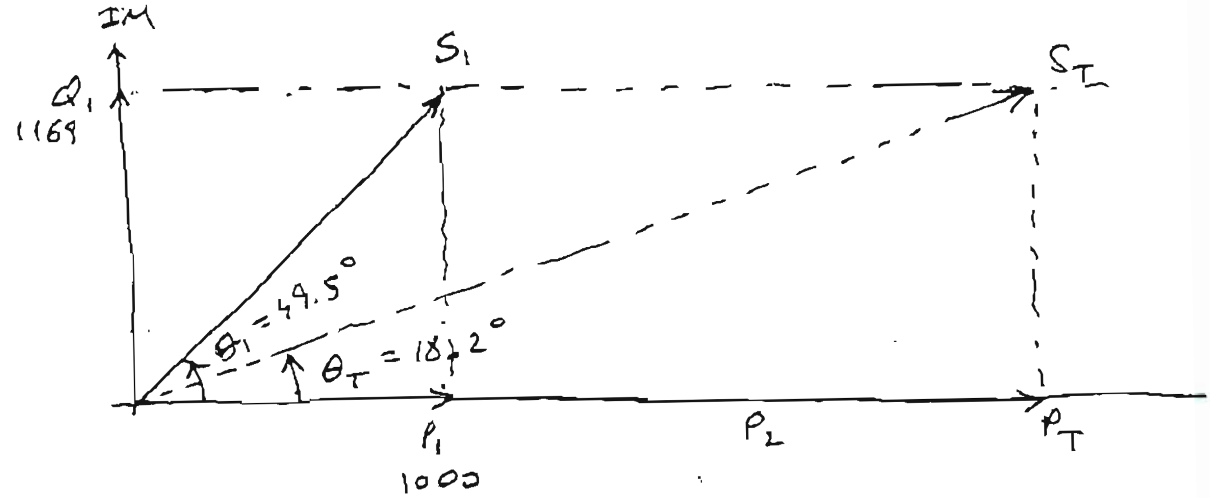
Ex: A load draws 1 kW under a PF = 0.65 inductive from a source of $\vec{V}_e = 220\angle 0^\circ$ Vrms, f = 50 Hz. What should be done to raise the PF to 0.95 inductive?

$$S_1 = 1000 + j\,\frac{1000}{0.65} \cdot \sin(\cos^{-1} 0.65)$$

$$= 1000 + j\,1169$$

* To obtain PF = 0.95, we need to have $\theta = \cos^{-1} 0.95 = 18.2^\circ$.
There are 2 possible solutions :

i) Increase P_1 while keeping Q_1 constant.

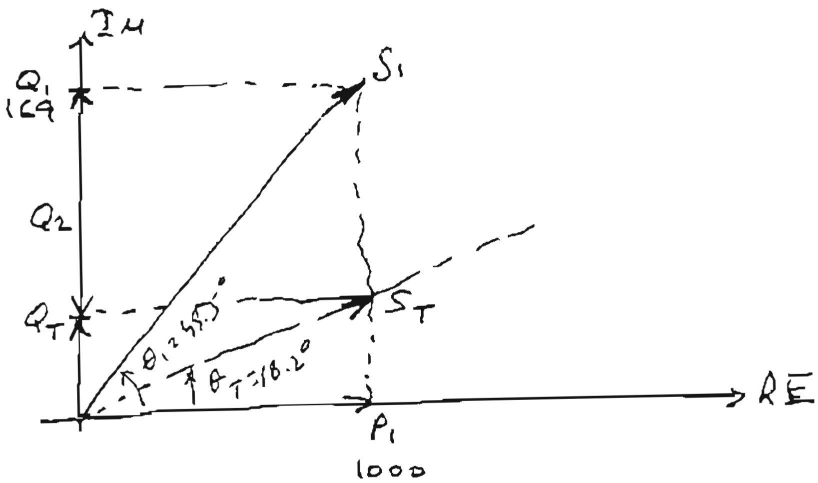


$$|S_T| = \frac{1169}{\sin 18.2°} = 3743$$

$$S_T = 3743\ (0.95) + j\,1169 = 3556 + j1169$$

Then $P_2 = 3556 - 1000 = 2556$ W. An resistive element has to be added to the system to use (waste ?) this power. So, this is not a good solution.

2) Decrease Q_1 while keeping P_1 constant.



$$S_T = 1000 + j\,\frac{1000}{0.95}\sqrt{1-0.95^2} = 1000 + j\,329$$

$$Q_2 = j329 - j1169 = -j840 = \vec{V}_2\,\vec{I}_2^{\,*}$$

$$\vec{I}_2^{\,*} = \frac{-j840}{220\angle 0°} = -j\,3.82 \longrightarrow \vec{I}_2 = j\,3.82\ A_{rms}$$

$$Z_2 = \frac{220\angle 0^\circ}{j3.82} = -j57.62\ \Omega$$

$$C = \frac{1}{377.50\ (57.62)} = 55.3\times10^{-6}\ F$$

$C = 55.3\ \mu F$ should be connected in parallel with the load. This operation is called "compensation". If the PF of a load is below 0.8, then this is penalized by charging a higher rate per kWh.

The current drawn from the line in the original case is

$$\vec{I}_1 = \left[\frac{S_1}{\vec{V}_e}\right]^* = \left[\frac{1000 + j1169}{220}\right]^* = 6.99\angle -49.5^\circ\ A_{rms}$$

The current after compensation is

$$\vec{I}_T = \left[\frac{S_T}{\vec{V}_e}\right]^* = \left[\frac{1000 + j329}{220}\right]^* = 4.76\angle -18.2^\circ\ A_{rms}$$

Let's assume a value for the resistance of the line the power is carried over, say $10\ \Omega$. Then

Original case: $P_{loss} = 10.(6.99)^2 = 489\ W$

After compensation: $P_{loss} = 10\ (4.76)^2 = 227\ W$

As we see, to do a certain job we are using 1000 W, but causing a power loss of 489 W in the first case, while to do the same job, we will cause a power loss of 227 W after compensation. This is less than half the loss caused originally.