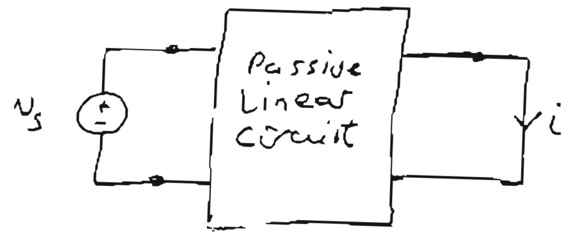
PHASOR CONCEPT



Let $v_s(t) = V_m \cos(\omega t + \alpha)$, then from our previous example we know that $i(t) = I_m \cos(\omega t + \beta)$. That is, a sinusoidal forcing function generates a sinusoidal response.

Let's shift the forcing function by $90°$:

$$V_m \cos(\omega t + \alpha - 90°) = V_m \sin(\omega t + \alpha)$$

Then the response will also be shifted by $90°$:

$$I_m \cos(\omega t + \beta - 90°) = I_m \sin(\omega t + \beta)$$

If we multiply the source by a constant K, then the response is also multiplied by K.

$$K V_m \sin(\omega t + \alpha) \longrightarrow K I_m \sin(\omega t + \beta)$$

Let $K = \sqrt{-1} = j$

$$j V_m \sin(\omega t + \alpha) \longrightarrow j I_m \sin(\omega t + \beta)$$

Then, using superposition, if we apply

$$V_m \cos(\omega t + \alpha) + j V_m \sin(\omega t + \alpha)$$

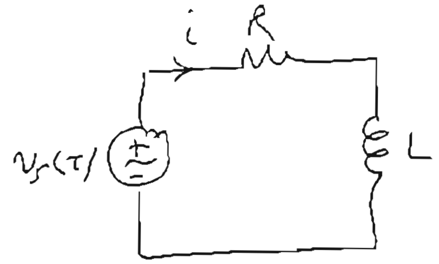
the response will be

$$I_m \cos(\omega t + \beta) + j I_m \sin(\omega t + \beta)$$

Using Euler's Identity $e^{jx} = \cos x + j \sin x$,

$$V_m e^{j(\omega t + \alpha)} \longrightarrow I_m e^{j(\omega t + \beta)}$$

A complex forcing function generates a complex response of which the real part is the response to the real part of the complex forcing function, the imaginary part is the response to the imaginary part of the complex forcing function.



$v_s(t) = V_m \cos(\omega t)$

Then the complex source is

$V_m e^{j\omega t}$

and the complex response will be

$I_m e^{j(\omega t+\theta)}$

$$Ri + L\frac{di}{dt} = V_s(t)$$

$$R I_m e^{j(\omega t+\theta)} + L\frac{d}{dt}\left[I_m e^{j(\omega t+\theta)}\right] = V_m e^{j\omega t}$$

$$R I_m e^{j(\omega t+\theta)} + j\omega L\, I_m e^{j(\omega t+\theta)} = V_m e^{j\omega t}$$

(1) $$R I_m e^{j\theta} + j\omega L\, I_m e^{j\theta} = V_m$$

$$I_m e^{j\theta} = \frac{V_m}{R+j\omega L} = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\, e^{j\left(-\tan^{-1}\frac{\omega L}{R}\right)}$$

and the real part of this expression, after multiplying by $e^{j\omega t}$, is

$$i'(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

* Note that, using the complex source, our integro differential equations, became simple algebraic equations.

In equation (1), after $e^{j\omega t}$'s cancelled, what is left in each term is of type

$I_m e^{j\theta}$ or $V_m e^{j\alpha}$

related to forcing function or response. These are complex number, vectors, in fact what we call as "phase vectors" or briefly "phasors". We will indicate them as

$$\vec{I} = I_m e^{j\theta} \quad \text{or} \quad \vec{V} = V_m e^{j\alpha}$$

Another way of showing these phasor is

$$\vec{I} = I_m \angle \theta \quad \text{or} \quad \vec{V} = V_m \angle \alpha$$

Ex

$$\vec{I} = 6 \angle -72^\circ = 6 e^{-j72^\circ} \rightarrow 6 e^{j(\omega t - 72^\circ)}$$

$\downarrow$ RE

$$i(t) = 6 \cos(\omega t - 72^\circ)$$

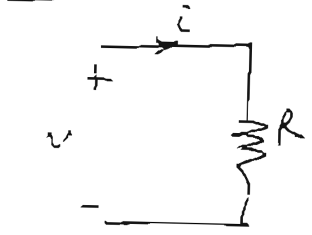
this transformation can directly be done by just inserting the amplitude and phase angle into the general expression of the time function $i(t) = I_m \cos(\omega t + \alpha)$.

Rewriting eq. (1) again:

$$R\vec{I} + j\omega L \vec{I} = \vec{V} \quad \text{and} \quad \vec{I} = \frac{\vec{V}}{R + j\omega L}$$

Phasor Relationship of R, L, and C:

Resistance:



$$v = Ri$$

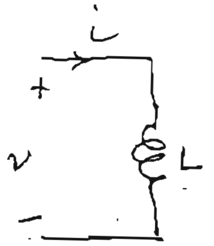
$$V_m e^{j(\omega t + \alpha)} = R \cdot I_m e^{j(\omega t + \beta)}$$

$$V_m e^{j\alpha} = R I_m e^{j\beta}$$

$$\vec{V} = R \vec{I}$$

Note that, since R is real, $\alpha = \beta$, that is the current and voltage of a resistance are at the same phase.

Inductance:



$$v = L \frac{di}{dt}$$

$$V_m e^{j(\omega t + \alpha)} = L \frac{d}{dt} \left[I_m e^{j(\omega t + \beta)} \right]$$

$$= j\omega L \cdot I_m e^{j(\omega t + \beta)}$$

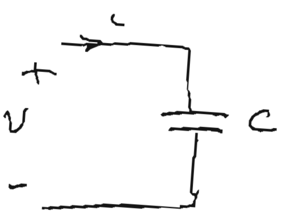
$$V_m e^{j\alpha} = j\omega L \cdot I_m e^{j\beta}$$

$$\vec{V} = j\omega L \cdot \vec{I}$$ where $Z_L = j\omega L$ (Ω) is called the impedance of an inductor.

* Since j implies $90°$ rotation CCW, current is lagging the voltage by $90°$ in a pure inductor.

* Note that Z_L is a function of ω.

Capacitance:



$$i = C \frac{dv}{dt}$$

$$I_m e^{j(\omega t+\beta)} = C \frac{d}{dt}\left[V_m e^{j(\omega t+\alpha)}\right]$$

$$= j\omega C \cdot V_m e^{j(\omega t+\alpha)}$$

$$I_m e^{j\beta} = j\omega C \cdot V_m e^{j\alpha}$$

$$\vec{I} = j\omega C \cdot \vec{V}$$

or $\vec{V} = \dfrac{1}{j\omega C} \vec{I}$ where $Z_C = \dfrac{1}{j\omega C} = -j\dfrac{1}{\omega C}$.

* $-j$ implies $90°$ rotation CW, current is leading the voltage by $90°$ in a pure capacitor.
* Note that Z_C is a function of ω.

Kirchhoff's Laws Using Phasors:

Voltage law for a loop is

$$v_1(t) + v_2(t) + \ldots + v_N(t) = 0$$

$$V_{m1} e^{j(\omega t+\theta_1)} + V_{m2} e^{j(\omega t+\theta_2)} + \ldots + V_{mN} e^{j(\omega t+\theta_N)} = 0$$

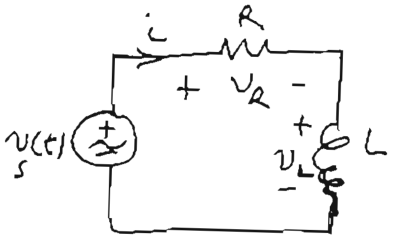
$$V_{m1} e^{j\theta_1} + V_{m2} e^{j\theta_2} + \ldots + V_{mN} e^{j\theta_N} = 0$$

$$\vec{V_1} + \vec{V_2} + \ldots + \vec{V_N} = 0$$

Same thing can be done for the current law.

Hence Kirchhoff's Laws are valid for phasors also. Then, all the methods derived based on these laws are valid when used with phasors. Here, we have to be careful though to use $\vec{V}-\vec{I}$ relations of the elements we obtained as the Ohm's Law.

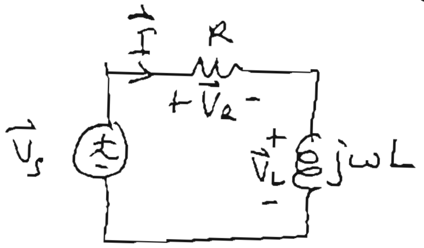
$$Z_R = R, \quad Z_L = j\omega L, \quad Z_C = -j\frac{1}{\omega C}$$



$$v_s(t) = v_R(t) + v_L(t)$$

$$\vec{V}_s = \vec{V}_R + \vec{V}_L$$

$$\vec{V}_s = R\vec{I} + j\omega L \cdot \vec{I} \longrightarrow \vec{I} = \frac{\vec{V}_s}{R+j\omega L} \quad (2)$$



and for $v_s(t) = V_m \cos\omega t$

$$\vec{V}_s = V_m \angle 0^\circ$$

$$\vec{I} = \frac{V_m \angle 0^\circ}{R+j\omega L} = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \angle -\tan^{-1}\frac{\omega L}{R}$$

and

$$i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

* Note that eq. (2) does not contain t, but it contains ω. Actually we are in the frequency domain. Hence impedances are in the frequency domain and we are working in the frequency domain with the phasors. Only at the end we transform our phasors to time domain.

* Note that impedance is a complex number and part of frequency domain, but not a phasor and it does not transform to time domain as phasors do.

Series and parallel combination of impedances:

Since impedance has a unit of ohm, their combinations can be obtained similar to resistance combinations.

Series : $Z_{eq} = Z_1 + Z_2 + \cdots + Z_k$

Parallel : $\dfrac{1}{Z_{eq}} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \cdots + \dfrac{1}{Z_k}$

For two elements

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Ex Let $L = 5\,H$, $C = 0.1\,F$. If $\omega = 10$ rad/s,

$$Z_L = j\omega L = j\,10.5 = j50\,\Omega, \quad Z_C = -j\frac{1}{\omega C} = -j\frac{1}{10(0.1)} = -j1$$

If series connected : $Z_{eq} = j50 - j1 = j49\,\Omega$

If parallel connected $Z_{eq} = \dfrac{(j50)(-j1)}{j50 - j1} = \dfrac{50}{j49} = -j\dfrac{50}{49}\,\Omega$

If we change ω, these values will change since impedances will change. Let's say $\omega = 1$ rad/s,

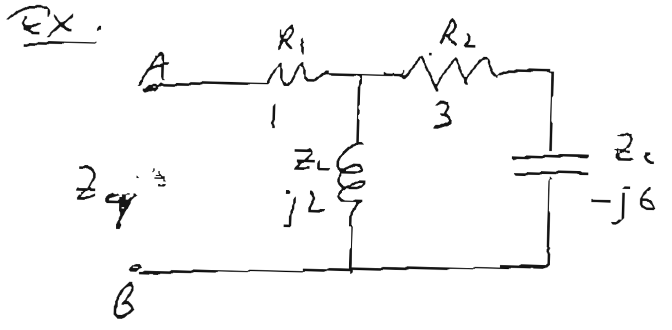
$$Z_L = j.1.5 = j5\,\Omega, \quad Z_C = -j\frac{1}{1(0.1)} = -j10$$

If series connected : $Z_{eq} = j5 - j10 = -j5\,\Omega$

If parallel connected : $Z_{eq} = \dfrac{(j5)(-j10)}{j5 - j10} = \dfrac{50}{-j5} = j10\,\Omega$

Note that, for series connection, the first example indicates an equivalent inductance since the sign of the impedance is +, while in the second example, equivalent impedance corresponds to a capacitor since it has a − sign.

Ex.



$Z_a = 3 - j6$

$$Z_b = \frac{(3-j6)(j2)}{3-j6+j2} = \frac{12+j6}{3-j4}$$

$$= \frac{(12+j6)(3+j4)}{3^2+4^2} = \frac{36+j48+j18-2}{25}$$

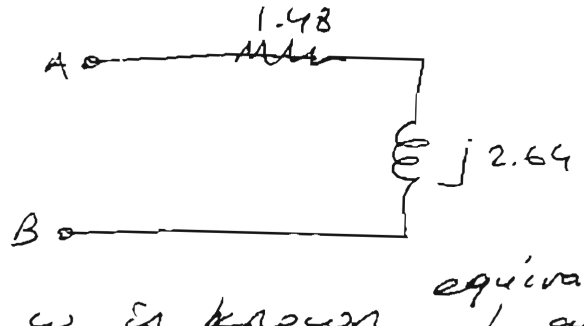
$$= \frac{12+j66}{25} = 0.48+j2.64$$

$$Z_{eq} = 1 + Z_b = 1 + 0.48 + j2.64 = 1.48 + j2.64 \ \Omega$$

$$= 3.03 \angle 60.7^\circ \ \Omega$$

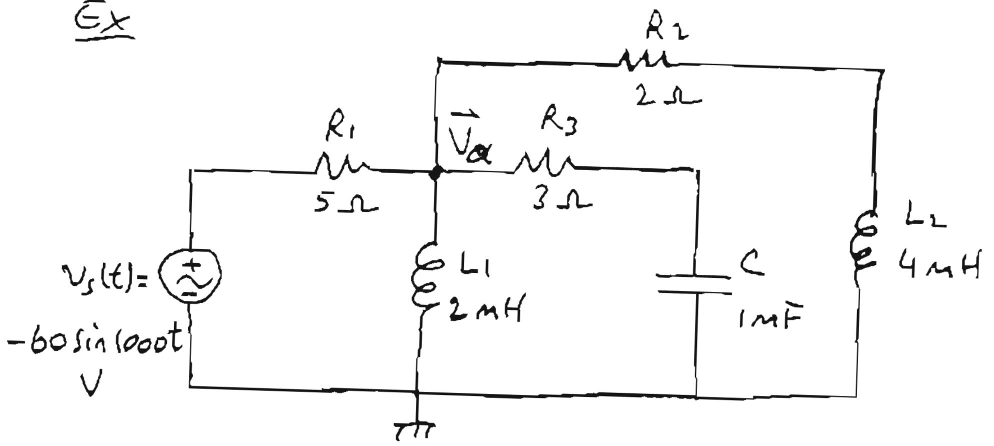
* Impedances can be expressed either in polar or rectangular form
* When impedances are expressed in rectangular form, the real part indicates the resistive component, the imaginary part including the sign indicates the reactive component

Hence, 1.48 Ω is the resistive component, 2.64 is the reactive component of the equivalent impedance. Then



* If ω is known, equivalent L and C values can be calculated.

Ex



$$Z_{L1} = j \cdot 10^3 \cdot 2 \cdot 10^{-3} = j2 \ \Omega$$
$$Z_{L2} = j \cdot 10^3 \cdot 4 \cdot 10^{-3} = j4 \ \Omega$$
$$Z_C = -j \frac{1}{10^3 \cdot 1 \cdot 10^{-3}} = -j1 \ \Omega$$
$$\vec{V}_s = -60(-j) = j60 \ V$$

$$\frac{\vec{V}_a - j60}{5} + \frac{\vec{V}_a}{j2} + \frac{\vec{V}_a}{3-j1} + \frac{\vec{V}_a}{2+j4} = 0$$

$$\frac{\vec{V}_a - j60}{5} - j\frac{\vec{V}_a}{2} + \frac{(3+j1)\vec{V}_a}{10} + \frac{(2-j4)\vec{V}_a}{20} = 0$$

$$4\vec{V}_a - j240 - j10\vec{V}_a + (6+j2)\vec{V}_a + (2-j4)\vec{V}_a = 0$$

$$(12 - j12)\vec{V}_a = j240$$

$$\vec{V}_a = \frac{j240}{12-j12} = \frac{j20}{1-j1} = j10(1+j1) = -10+j10 \ V$$

$$\vec{V}_{R1} = j60 - (-10+j10) = 10+j50 \ V \qquad \vec{I}_{R1} = \frac{10+j50}{5} = 2+j10 \ A$$

$$\vec{V}_{L1} = -10+j10 \ V \qquad \vec{I}_{L1} = \frac{-10+j10}{j2} = 5+j5 \ A$$

and

$$\vec{I}_{R3} = \vec{I}_C = \frac{-10+j10}{3-j1} = (-1+j1)(3+j1) = -4+j2 \ A$$

$$\vec{V}_{R3} = 3(-4+j2) = -12+j6 \ V$$

$$\vec{V}_C = -j1(-4+j2) = 2+j4 \ V$$

and

$$\vec{I}_{R2} = \vec{I}_{L2} = \frac{-10+j10}{2+j4} = \frac{(-1+j1)(2-j4)}{2} = 1+j3 \ A$$

$$\vec{V}_{R2} = 2(1+j3) = 2+j6 \ V$$

$$\vec{V}_{L2} = j4(1+j3) = -12+j4 \ V$$

If, for example, we are interested in the source current

$$\vec{I}_s = 2+j10 = \sqrt{2^2+10^2} \ \angle \tan^{-1}\frac{10}{2} = 10.2 \angle 78.7^\circ \ A$$

$$i_s(t) = 10.2 \cos(1000t + 78.7^\circ) \ A$$

Admittance: Admittance is the ratio of the phasor current to phasor voltage. Then

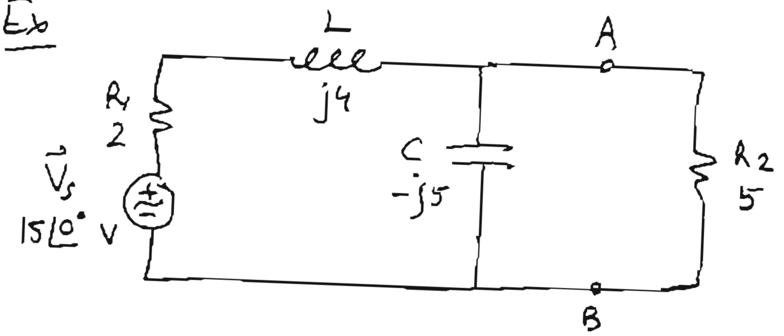
$$Y = \frac{\vec{I}}{\vec{V}} = \frac{1}{\vec{V}/\vec{I}} = \frac{1}{Z} \qquad (\text{mho},\ \mho,\ \text{Siemens (S)})$$

The real part of admittance is called conductance (G), the imaginary part susceptance (B).

$$Y = G + jB = \frac{1}{Z} = \frac{1}{R+jX} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2}$$

* Note that $G \neq \frac{1}{R}$ and $B \neq \frac{1}{X}$.

Ex

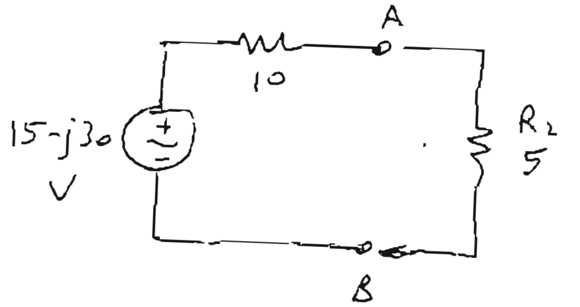


Find the Thevenin equivalent of the circuit between A and B seen by the resistor R_2.

$$\vec{V}_{oc} = \frac{15\angle 0^\circ}{2+j4-j5}\cdot(-j5) = \frac{-j75}{2-j1} = \frac{-j75\,(2+j1)}{5} = -j15\,(2+j1)$$

$$\vec{V}_{oc} = 15 - j30 \ \text{V}$$

$$Z_{Th} = \frac{(2+j4)(-j5)}{2+j4-j5} = \frac{20-j10}{2-j1} = 10$$



$$\vec{I}_{R2} = \frac{15-j30}{10+5} = 1-j2 \ \text{A}$$

$$\vec{V}_{R2} = 5\,(1-j2) = 5-j10 \ \text{V}$$