Linearity and the Superposition Theorem

We have already talked about linear elements and linear circuits.

Any element with a linear current-voltage (i-v) relationship is linear, including dependent sources.

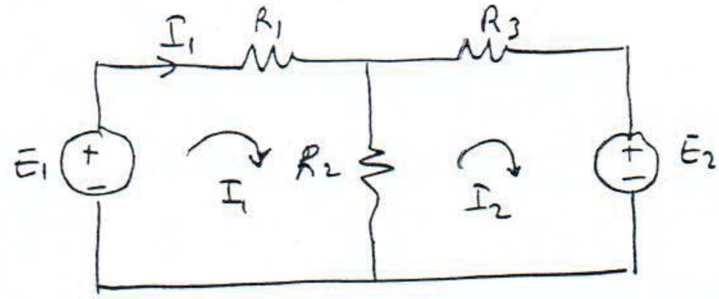
An important result of linearity is superposition.

Superposition theorem: In any linear circuit containing several independent sources, the voltage or current in any element can be found by adding algebraicly the currents or voltages caused by each independent source acting alone while all the other independent sources are killed.

Note: Killing a voltage source means short-circuiting it, killing a current source means open-circuiting it.

Note: Only the independent sources are considered and killed, dependent sources are kept as they are.

Let's look at the circuit below, and obtain the solution for, let's say, I_1, the current throug the resistor R_1.



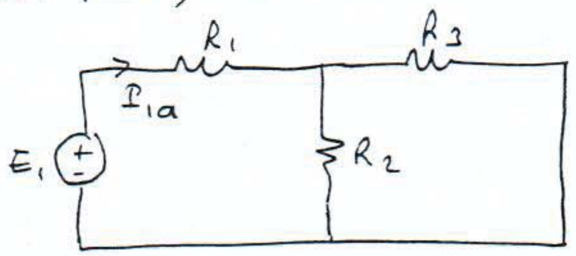
$$(R_1+R_2)I_1 - R_2 I_2 = E_1$$
$$-R_2 I_1 + (R_2+R_3) I_2 = -E_2$$
$$I_2 = \frac{(R_1+R_2)I_1 - E_1}{R_2}$$

$$-R_2 I_1 + (R_2+R_3)\,\frac{(R_1+R_2)I_1 - E_1}{R_2} = -E_2$$

$$\left(-R_2^2 + R_1R_2 + R_2^2 + R_1R_3 + R_2R_3\right) I_1 - (R_2+R_3)E_1 = -R_2E_2$$

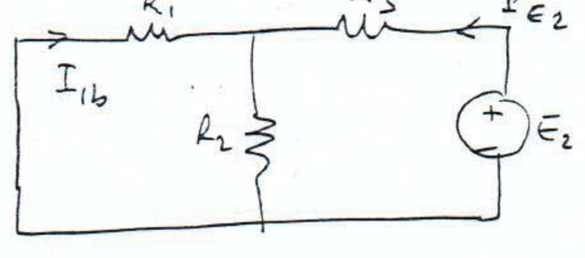
$$I_1 = \underbrace{\frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}\,E_1}_{\text{contribution of } E_1} - \underbrace{\frac{R_2}{R_1R_2+R_1R_3+R_2R_3}\,E_2}_{\text{contribution of } E_2}$$

In fact, with $E_2 = 0$:



$$I_{1a} = \frac{E_1}{R_1 + \frac{R_2R_3}{R_2+R_3}} = \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}\,E_1$$

with $E_1 = 0$:

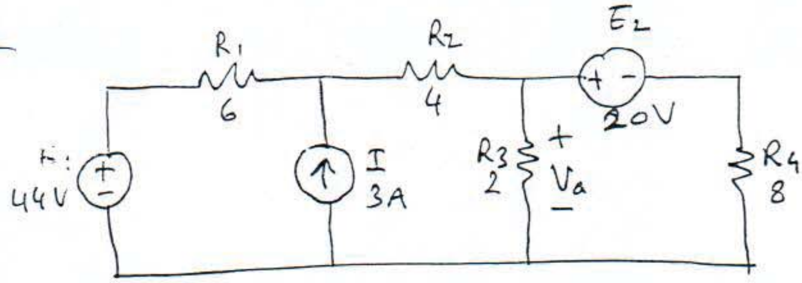


$$I_{E2} = \frac{E_2}{R_3 + \frac{R_1R_2}{R_1+R_2}} = \frac{R_1+R_2}{R_1R_2+R_1R_3+R_2R_3}\,E_2$$

$$I_{1b} = -\frac{R_2}{R_1+R_2}\,I_{E2} = -\frac{R_2}{R_1R_2+R_1R_3+R_2R_3}\,E_2$$

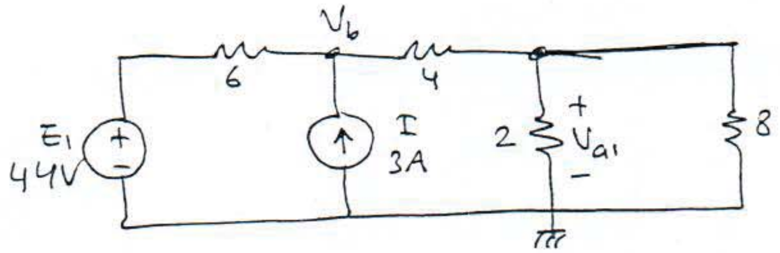
Note: We do not have to analyze the circuits for each source separately. Sources may be grouped in any convenient way if desired. As long as all the sources are included in the analys only once, correct solution will be obtained.

Ex



$V_a = ?$

i, with E_1 and I present:



$$\frac{V_b - 44}{6} - 3 + \frac{V_b - V_{a1}}{4} = 0$$

$$\frac{V_{a1} - V_b}{4} + \frac{V_{a1}}{2} + \frac{V_{a1}}{8} = 0$$

$$2V_b - 88 - 36 + 3V_b - 3V_{a1} = 0$$

$$2V_{a1} - 2V_b + 4V_{a1} + V_{a1} = 0$$

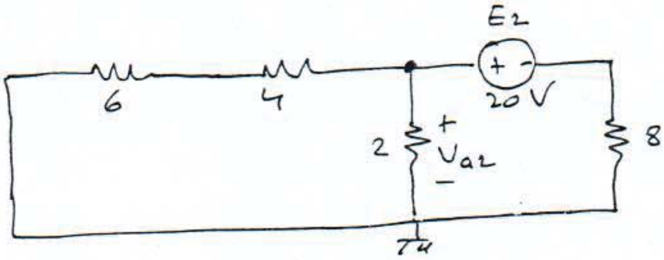
$$5V_b - 3V_{a1} = 124$$

$$-2V_b + 7V_{a1} = 0 \rightarrow V_b = 3.5V_a$$

$$17.5\,V_{a1} - 3V_{a1} = 124$$

$$V_{a1} = \frac{124}{14.5} = \frac{248}{29}\ \text{V}$$

ii, with E_2 present:



$$\frac{V_{a2}}{6+4} + \frac{V_{a2}}{2} + \frac{V_{a2} - 20}{8} = 0$$

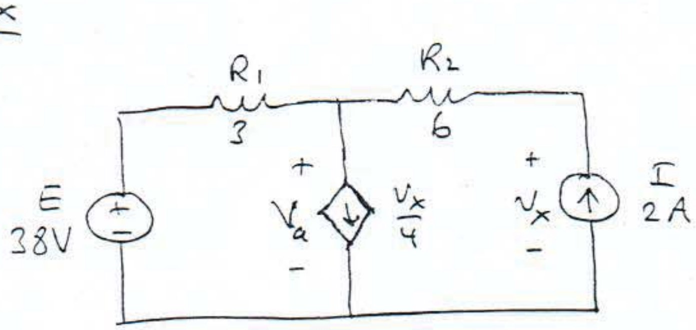
$$4V_{a2} + 20V_{a2} + 5V_{a2} - 100 = 0$$

$$V_{a2} = \frac{100}{29}\ \text{V}$$

Hence

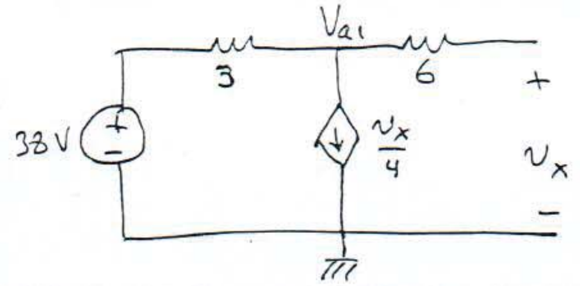
$$V_a = V_{a1} + V_{a2} = \frac{248}{29} + \frac{100}{29} = \frac{348}{29} = 12\,\text{V}$$

Ex



Find V_a using superposition

With E only $(I=0)$:



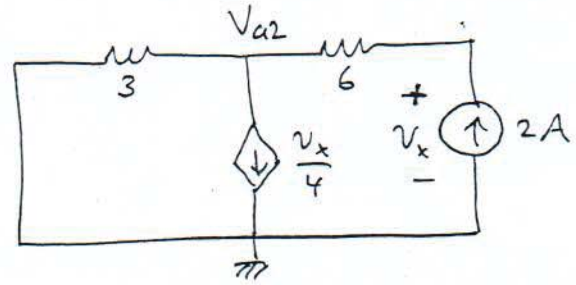
$$v_x = V_{a1}$$

$$\frac{V_{a1}-38}{3} + \frac{v_x}{4} = 0$$

$$4V_{a1} - 152 + 3V_{a1} = 0$$

$$V_{a1} = \frac{152}{7}\ \text{V}$$

With I only $(E=0)$:



$$\frac{V_{a2}}{3} + \frac{v_x}{4} - 2 = 0$$

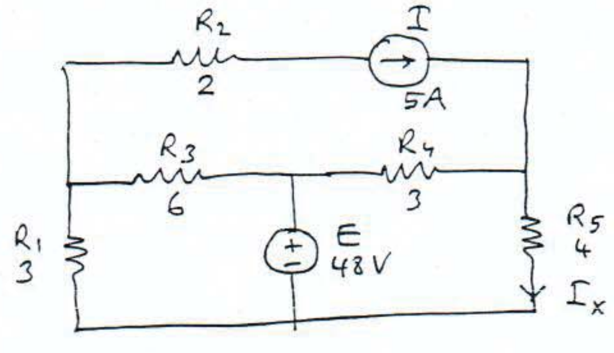
$$v_x = V_{a2} + 6\times 2$$

$$4V_{a2} + 3\,(V_{a2}+12) - 24 = 0$$

$$V_{a2} = -\frac{12}{7}\ \text{V}$$

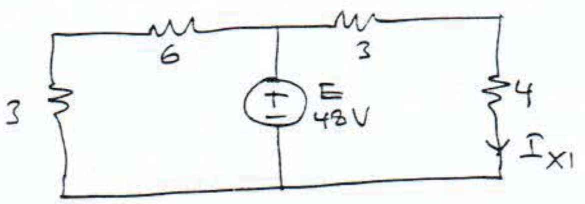
Hence $V_a = V_{a1} + V_{a2} = \frac{152}{7} + \left(-\frac{12}{7}\right) = \frac{140}{7} = 20\ \text{V}$

Ex



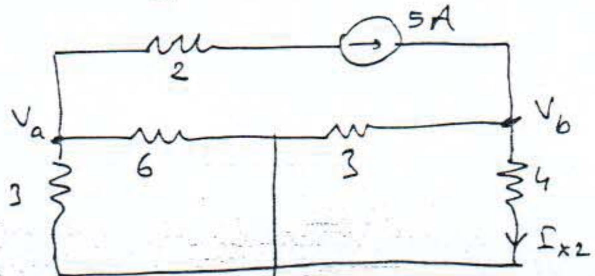
Find I_x using superposition.

With E only:



$$I_{x1} = \frac{48}{3+4} = \frac{48}{7}\ \text{A}$$

With I only:



$$\frac{V_b}{3} + \frac{V_b}{4} - 5 = 0$$

$$4V_b + 3V_b = 60 \rightarrow V_b = \frac{60}{7}$$

$$I_{x2} = \frac{15}{7}\ \text{A}$$

$$I_x = \frac{48}{7} + \frac{15}{7} = \frac{63}{7} = 9\ \text{A}$$