MAGNETIC CIRCUITS

The study of *magnetic circuits* is important in the study of energy systems since the operation of key components such as transformers and rotating machines (DC machines, induction machines, synchronous machines) can be characterized efficiently using magnetic circuits. Magnetic circuits, which characterize the behavior of the magnetic fields within a given device or set of devices, can be analyzed using the circuit analysis techniques defined for electric circuits.

The quantities of interest in a magnetic circuit are the vector magnetic field $\boldsymbol{H}$ (A/m), the vector magnetic flux density $\boldsymbol{B}$ (T = Wb/m^2) and the total magnetic flux ψ_m (Wb). The vector magnetic field and vector magnetic flux density are related by

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_r \mu_o \boldsymbol{H}$$

where μ is defined as the total permeability (H/m), μ_r is the relative permeability (unitless), and $\mu_o = 4\pi \times 10^{-7}$ H/m is the permeability of free space. The total magnetic flux through a given surface S is found by integrating the normal component of the magnetic flux density over the surface

$$\psi_m = \iint_S \boldsymbol{B} \cdot d\boldsymbol{s}$$

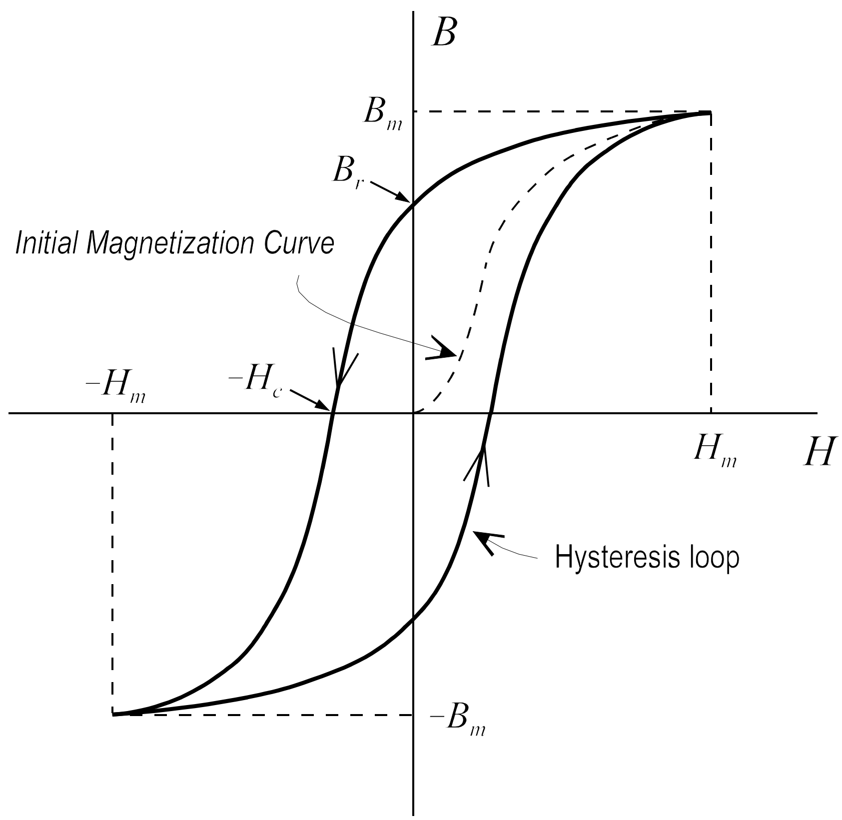
where the vector differential surface is given by $d\boldsymbol{s} = \boldsymbol{a}_n ds$ and where $\boldsymbol{a}_n$ defines a unit vector normal to the surface S.

The relative permeability is a measure of how much magnetization occurs within the material. There is no magnetization in free space (vacuum) and negligible magnetization in common conductors such as copper and aluminum. These materials are characterized by a relative permeability of unity (μ_r =1).

There are certain magnetic materials with very high relative permeabilities that are commonly found in components of energy systems. These materials (iron, steel, nickel, cobalt, etc.), designated as *ferromagnetic materials*, are characterized by significant magnetization. Ferromagnetic materials can be thought of as efficient conductors of magnetic fields. The relative permeabilities of ferromagnetic materials can range from a few hundred to a few thousand. Ferromagnetic materials are highly *nonlinear*. That is, the relative permeability is not a constant, but depends on the magnitude of the magnetic field for a given problem. Thus, the relationship between the magnetic field and the magnetic flux density in a nonlinear medium can be written as

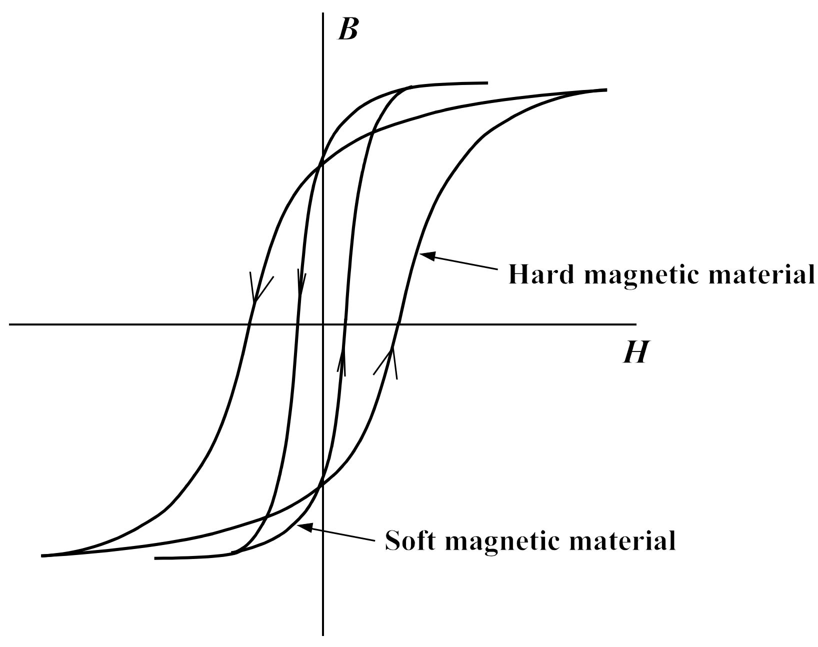
$$\boldsymbol{B} = \mu(H)\boldsymbol{H}$$

The characteristics of ferromagnetic materials are typically presented using the *B-H* curve, a plot of the magnetic flux density *B* in the material due to a given applied magnetic field *H*. The *B-H* curve shows the *initial magnetization curve* along with a curve known as a *hysteresis loop*.



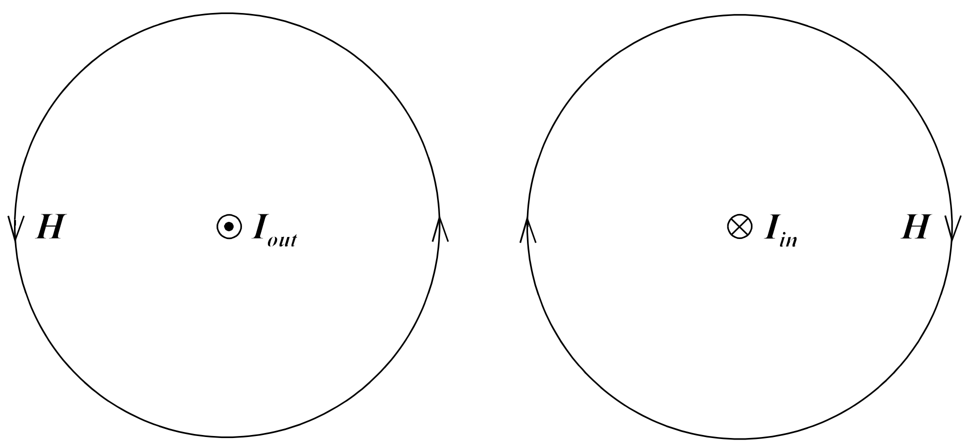
The initial magnetization curve shows the magnetic flux density that would result when an increasing magnetic field is applied to an initially unmagnetized material. An unmagnetized material is defined at the origin of the B-H curve (B=H=0) where there is no net magnetic flux given no applied field. As the magnetic field increases, at some point, all of the magnetic moments (current loops) within the material align themselves with the applied field and the magnetic flux density saturates (B_m). If the magnetic field is then cycled between the saturation magnetic field value in the forward and reverse directions ($\pm H_m$), the hysteresis loop results. The response of the material to any applied field depends on the initial state of the material magnetization at that instant.

Two quantities of particular interest on the hystersis loop are the *retentivity* B_r (or residual flux density) and the *coercivity* H_c (or coercive force). Note that the retentivity is a measure of how much of the magnetic energy is retained by the material upon removal of the applied magnetic field. The higher the retentivity relative to the saturation level B_m, the more of the applied magnetic energy is stored in the material. The coercivity is related to the demagnetization of the material. Note that the smaller the coercivity, the closer this point is to the point of total demagnetization (the origin of the B-H curve). Materials with low coercivities can take less energy to demagnetize (sometimes called *soft* magnetic materials). Conversely, materials with high coercivities, are known as *hard* magnetic materials.

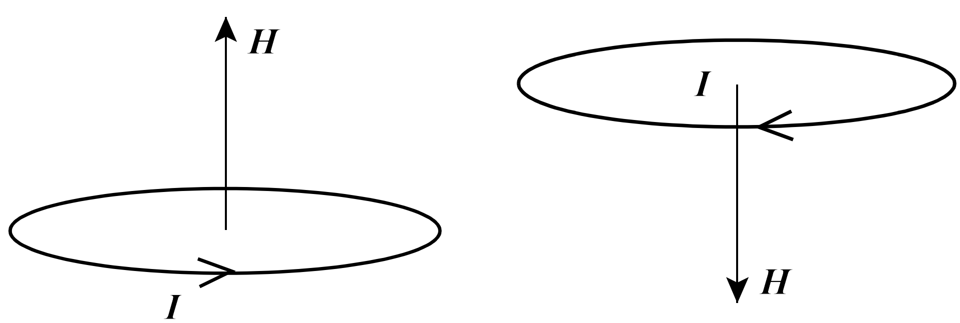


CURRENT AND MAGNETIC FIELD

The source of magnetic field is current. The magnetic field of a current-carrying wire will basically encircle the wire. The magnitude of the magnetic field grows smaller as distance from the wire increases. The direction of the magnetic field produced by a current-carrying conductor is determined by the *right-hand rule*. The magnetic field produced by a straight conductor is shown below. By placing the thumb of one's right hand in the direction of the current, the fingers wrap around the wire in the direction of the magnetic field.



The magnetic field of a current-carrying conductor can be concentrated by forming the conductor in the shape of a loop. The magnetic field direction through the loop can also be determined by the right hand rule. Aligning one's fingers in the direction of the loop current, the thumb defines the direction of the magnetic field through the loop.



The magnetic field of a current-carrying conductor can be further concentrated by forming multiple turn loops (coils). This, in effect, increases the magnetic field by a multiplication factor equal to the number of turns *N*.

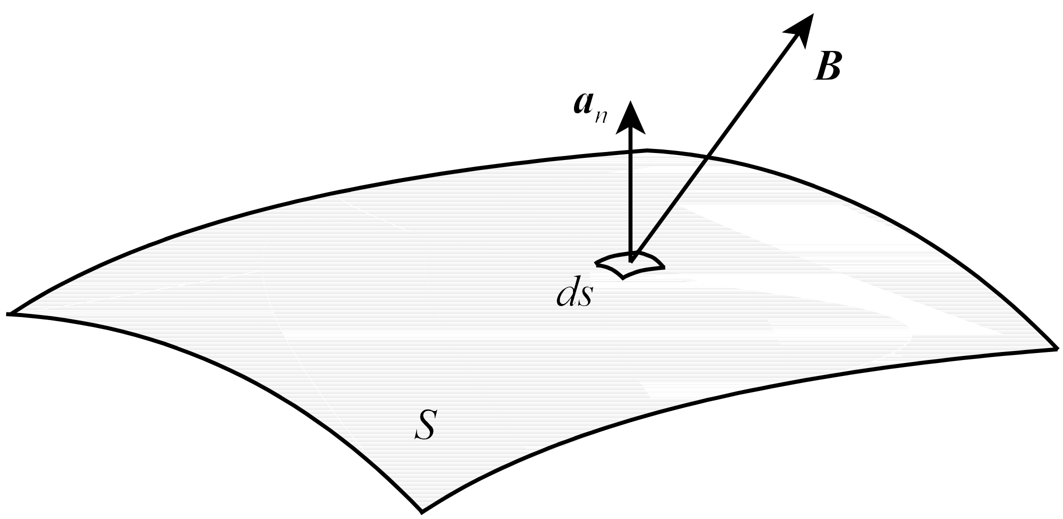
MAGNETIC FLUX DENSITY AND TOTAL MAGNETIC FLUX

The total magnetic flux passing through a given surface *S* is found by integrating the normal component of the magnetic flux density over the surface.

$$\psi_m = \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} = \iint_S B_n \, ds$$

where

$d\boldsymbol{s} = \boldsymbol{a}_n \, ds$
$\boldsymbol{a}_n$ − unit vector normal to the surface *S*
ds − differential surface element on *S*



$$\boldsymbol{B} \cdot d\boldsymbol{s} = \boldsymbol{B} \cdot \boldsymbol{a}_n \, ds = B_n \, ds$$

For the special case of a uniform magnetic flux density over the surface S, the integral for the total magnetic flux reduces to

$$\psi_m = \iint_S \boldsymbol{B} \cdot d\boldsymbol{s} = \iint_S B_n \, ds = B_n \iint_S ds = B_n A$$

where A is the total surface area of S. Thus, in regions of uniform magnetic flux density, the magnetic flux density is equal to the total magnetic flux ψ_m divided by the total area A.

$$B_n = \frac{\psi_m}{A} \qquad \text{(uniform flux density)}$$

The magnetic flux density is commonly assumed to be uniform within ferromagnetic materials.

AMPERE'S LAW

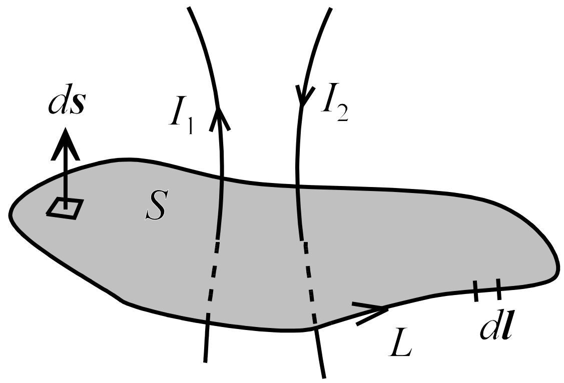
Ampere's law is the Maxwell equation that relates the magnetic field (flux) to the source of the magnetic field (current).

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \left[\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right] \cdot d\boldsymbol{s} \qquad \text{(Ampere's law)}$$

> ***Ampere's Law*** - The line integral of the magnetic field around a closed path equals the net current enclosed (conduction plus displacement current).

Note that the current direction is implied by the direction of the path L and the normal to the surface S according to the right hand rule. The displacement current is negligible at the commonly-used power frequencies of energy systems. Under this assumption, Ampere's law reduces to

$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} \approx \iint_S \boldsymbol{J} \cdot d\boldsymbol{s} = I_{enclosed} \qquad \text{(Ampere's law - low freq.)}$$



Thus, at low frequency, the line integral of the magnetic field around the closed path L yields the net conduction current passing through the surface S.

Example (Ampere's law / infinite-length line current)

Given a infinite-length line current I lying along the z-axis, use Ampere's law to determine the magnetic field by integrating the magnetic field around a circular path of radius ρ lying in the x-y plane.

From Ampere's law,

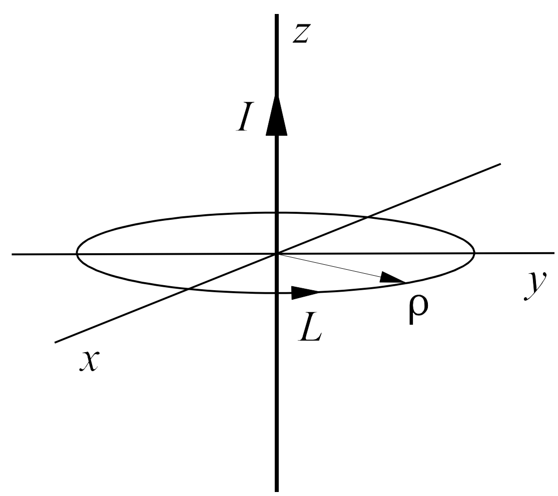
$$\oint_L \boldsymbol{H} \cdot d\boldsymbol{l} = \oint_L H_\phi \, dl = I$$

By symmetry, the magnetic field is uniform on the given path so that

$$H_\phi \oint_L dl = H_\phi (2\pi\rho) = I$$

or

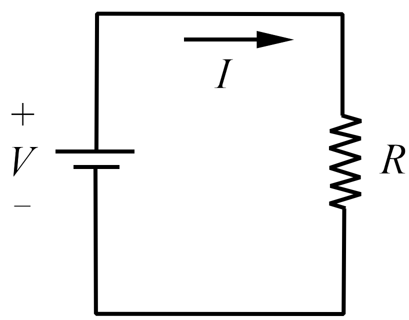
$$H_\phi = \frac{I}{2\pi\rho}$$

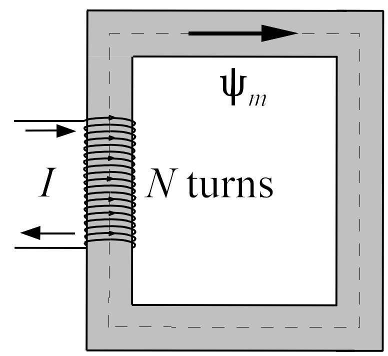


MAGNETIC CIRCUIT ANALYSIS

Magnetic field problems involving components such as current coils, ferromagnetic cores and air gaps can be solved as *magnetic circuits* according to the analogous behavior of the magnetic quantities to the corresponding electric quantities in an electric circuit.

Electric Circuit	Magnetic Circuit





$$V = IR \qquad\qquad \mathscr{F} = \psi_m \Re$$

V = electromotive force (V) [emf]

I = total current (A)

R = resistance (Ω)

$\mathscr{F}$ = magnetomotive force (A−turns) [mmf]

ψ_m = total magnetic flux (Wb)

$\Re$ = reluctance (H^{-1})

$$V = \oint \boldsymbol{E}\cdot d\boldsymbol{l} \qquad\qquad \mathscr{F} = \oint \boldsymbol{H}\cdot d\boldsymbol{l} = NI$$

$$I = \iint_S \boldsymbol{J}\cdot d\boldsymbol{s} \qquad\qquad \psi_m = \iint_S \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$R = \frac{l}{\sigma A} = \frac{1}{G} \qquad\qquad \Re = \frac{l}{\mu A} = \frac{1}{\wp}$$

G = conductance (S)

$\wp$ = permeance (H)

Given that *reluctance* in a magnetic circuit is analogous to resistance in an electric circuit, and permeability in a magnetic circuit is analogous to conductivity in an electric circuit, we may interpret the permeability of a medium as a measure of the resistance of the material to magnetic flux. Just as current in an electric circuit follows the path of least resistance, the magnetic flux in a magnetic circuit follows the path of least reluctance.

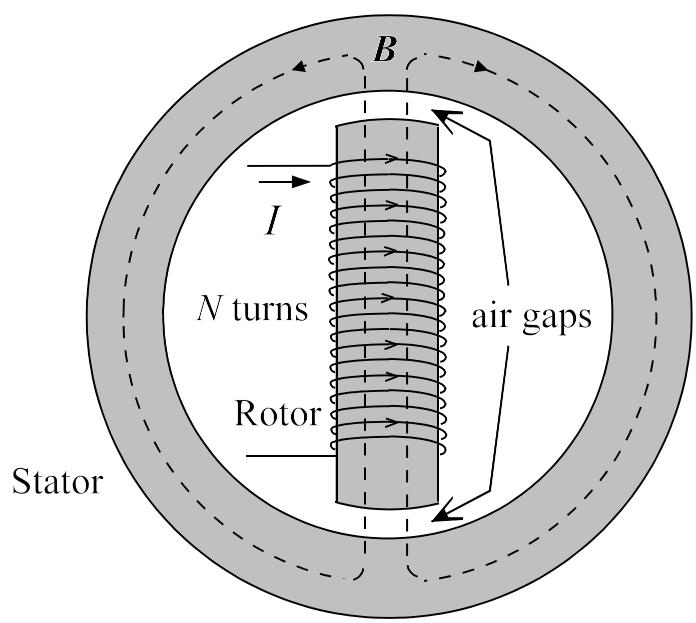
The ferromagnetic cores of transformers form closed loops (no air gaps). The ferromagnetic cores of rotating machinery must have air gaps in the resulting magnetic circuit. These air gaps typically represent a large portion of the overall magnetic circuit reluctance. In some cases of magnetic circuits with air gaps, we neglect the reluctance of the ferromagnetic core. Assuming the reluctance of the ferromagnetic core is zero is equivalent to assuming that the relative permeability of the core is infinite.

$$\Re = \frac{l}{\mu_o \mu_r A} \rightarrow 0 \quad \text{as} \quad \mu_r \rightarrow \infty$$

This is analogous to neglecting the resistance of conductors with very high conductivity in electric circuits.

<u>Example</u> (Magnetic circuit, synchronous machine)

The physical parameters of the synchronous machine are:

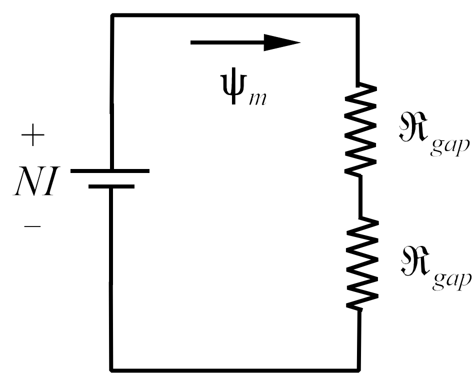


Air gap length $g = 1\text{cm}$
$I = 10$ A
$N = 1000$ turns
Rotor pole face area $A_r = 0.2\ \text{m}^2$

Assume that the rotor and the stator of the synchronous machine have negligible reluctance (infinite permeability) and neglect fringing.

(a.) Draw the magnetic circuit.
(b.) Determine the magnetomotive force.
(c.) Determine the reluctance of each air gap.
(d.) Determine the total magnetic flux in each air gap.
(e.) Determine the magnetic flux density in each air gap.

(a.)



(b.) $NI = (1000)(10) = 10^4$ A-turns

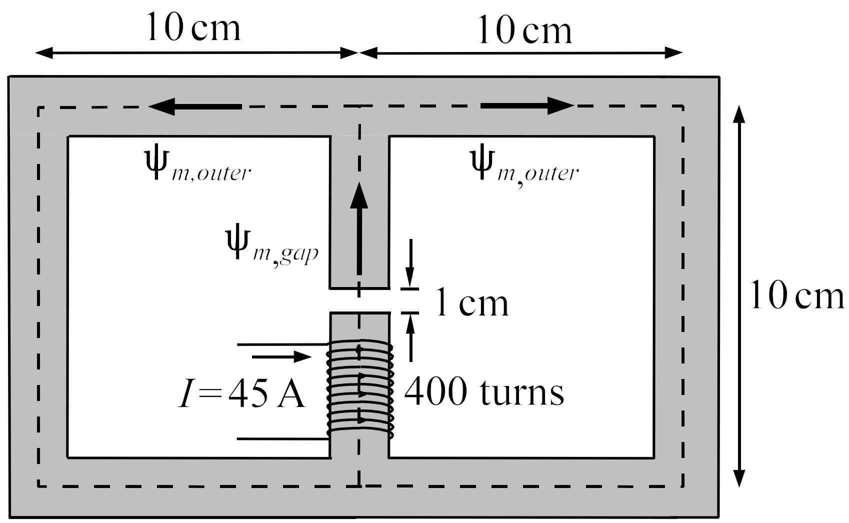
(c.) $$\Re_{gap} = \frac{l}{\mu A} = \frac{g}{\mu_o A_r} = \frac{0.01}{(4\pi \times 10^{-7})(0.2)} = 3.98 \times 10^4\ \text{H}^{-1}$$

(d.) $$\psi_m = \frac{NI}{2\Re_{gap}} = \frac{10^4}{(2)(3.98 \times 10^4)} = 0.126\ \text{Wb}$$

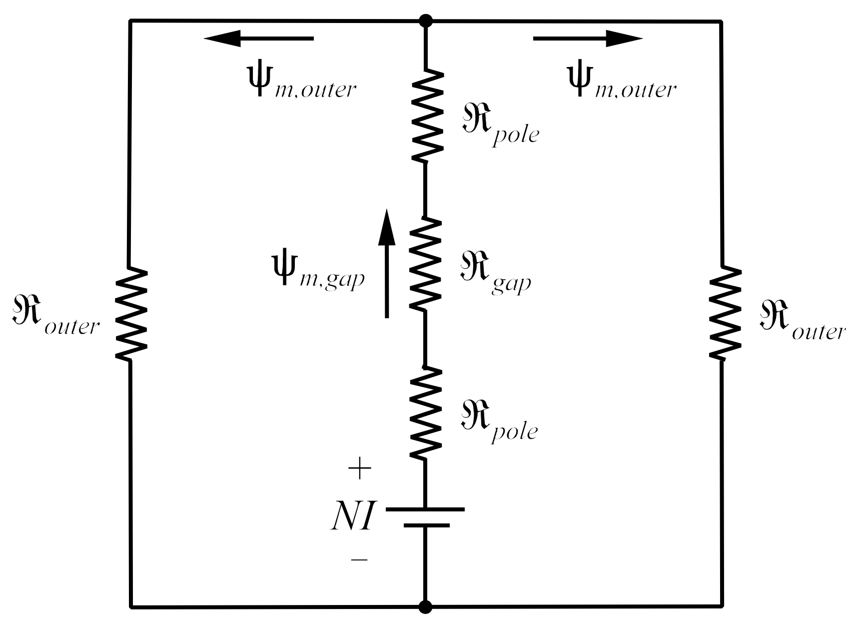
(e.) $$B = \frac{\psi_m}{A_r} = \frac{0.126}{0.2} = 0.630\ \text{T}$$

Example (Series/parallel magnetic circuits)

Determine the magnetic field in the air gap of the magnetic circuit shown below. The cross sectional area of all branches is 10 cm^2 and μ_r=50.



The equivalent electric circuit is



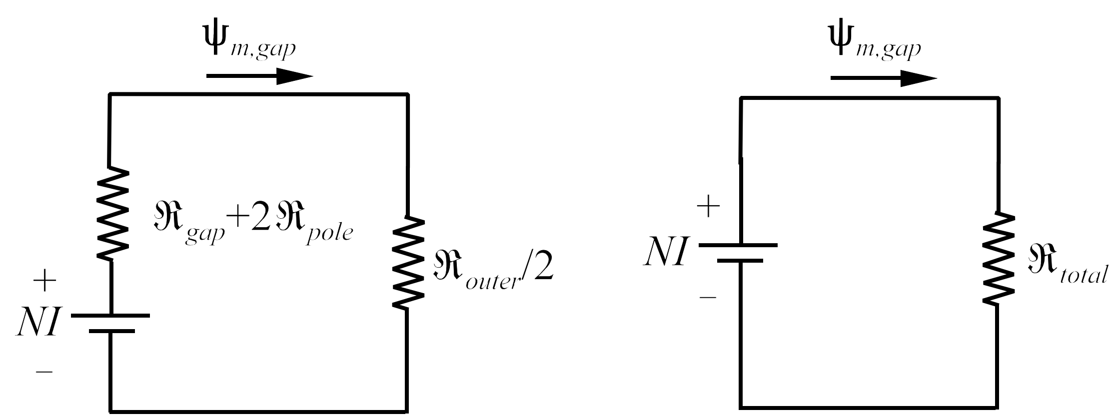
The reluctance components for the magnetic circuit are

$$\mathfrak{R}_{outer} = \frac{l_{outer}}{\mu A} = \frac{0.3}{50\mu_o(10^{-3})} = 4.77\times 10^6 \text{ H}^{-1}$$

$$\mathfrak{R}_{pole} = \frac{l_{pole}}{\mu A} = \frac{0.045}{50\mu_o(10^{-3})} = 7.16\times 10^5 \text{ H}^{-1}$$

$$\mathfrak{R}_{gap} = \frac{l_{gap}}{\mu_o A} = \frac{0.01}{\mu_o(10^{-3})} = 7.96\times 10^6 \text{ H}^{-1}$$

The equivalent circuit can be reduced to



$$\mathfrak{R}_{total} = \mathfrak{R}_{gap} + 2\,\mathfrak{R}_{pole} + \frac{\mathfrak{R}_{outer}}{2} = 11.78\times 10^6 \text{ H}^{-1}$$

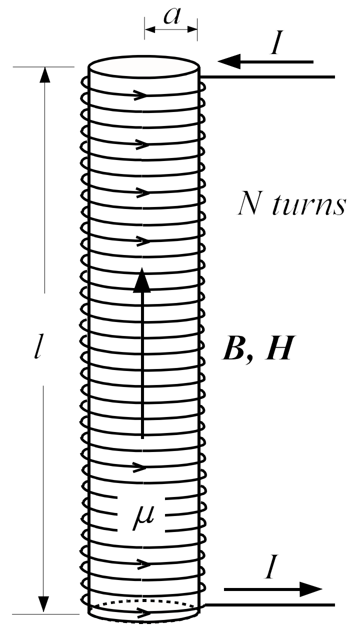
$$NI = \psi_{m,gap}\mathfrak{R}_{total} \quad\Rightarrow\quad \psi_{m,gap} = \frac{NI}{\mathfrak{R}_{total}} = \frac{400(45)}{11.78\times 10^6} = 1.53 \text{ mWb}$$

$$\psi_{m,gap} = B_{gap}A \quad\Rightarrow\quad B_{gap} = \frac{\psi_{m,gap}}{A} = \frac{1.53\times 10^{-3}}{10^{-3}} = 1.53 \text{ T}$$

$$H_{gap} = \frac{B_{gap}}{\mu_o} = \frac{1.53}{4\pi\times 10^{-7}} = 1.22 \text{ MA/m}$$

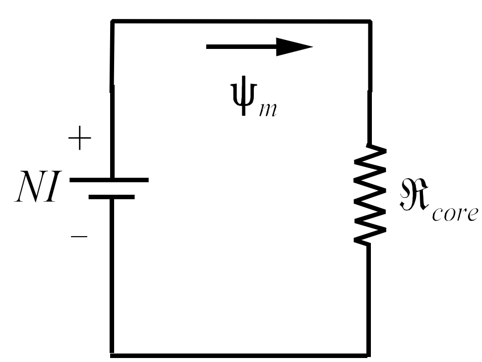
SOLENOIDS, TOROIDS, AND OTHER UNIFORM CORES

The equation for the magnetic flux density in any uniform ferromagnetic core device (solenoids, toroids, etc.) can be determined easily using the magnetic circuit technique. A uniform ferromagnetic core is one that has a uniform cross-section around the entire magnetic circuit.



The magnetomotive force for the long solenoid [length = l, radius = a, ($l \gg a$), core permeability = μ, number of turns = N, and current = I] is NI and is connected in series with the reluctance of the ferromagnetic core ($\Re_{core}$). Note that for the solenoid, the magnetic flux passes out of one end of the ferromagnetic core to the surrounding air before entering the opposite end of the core. Although the air is an inferior magnetic conductor in comparison to the core material, the magnetic flux has an infinite cross-sectional area of air to pass through. Thus, the reluctance of the surrounding air is, in effect, zero and the resulting magnetic circuit is simply the mmf in series with $\Re_{core}$. The resulting total magnetic flux in the solenoid is given by

$$\psi_m = \frac{NI}{\Re_{core}} = \frac{NI}{\dfrac{l}{\mu A}} = \frac{\mu NIA}{l} = \frac{\mu NI\pi a^2}{l}$$

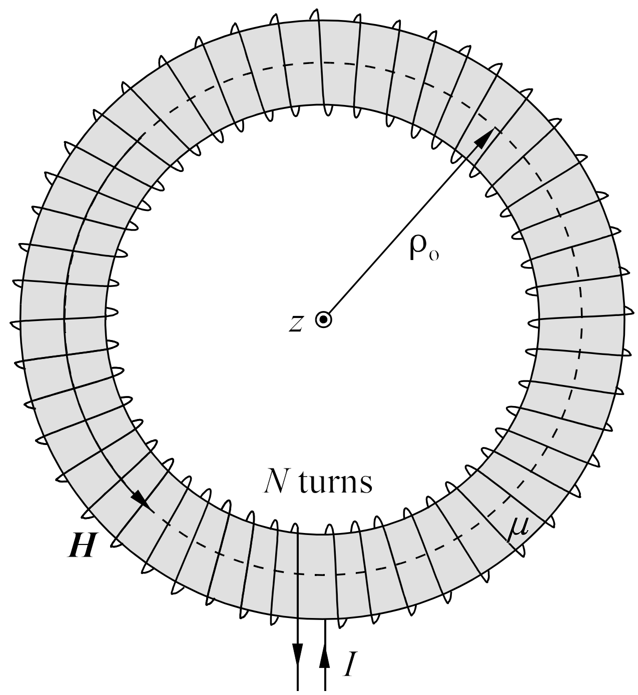


Assuming a uniform magnetic flux density, $\boldsymbol{B}$ is the total magnetic flux divided by the cross-sectional area of the magnetic core:

$$B = \frac{\psi_m}{A} = \frac{\mu NI}{l} \qquad \text{(long solenoid)}$$

The assumption that the solenoid is “long” is necessary in order to assume that the magnetic flux density is approximately uniform over the entire volume inside the solenoid. The magnetic flux density actually grows smaller near the ends of the long solenoid, but if $l \gg a$, then the fringing effects near the end of the solenoid become negligible.

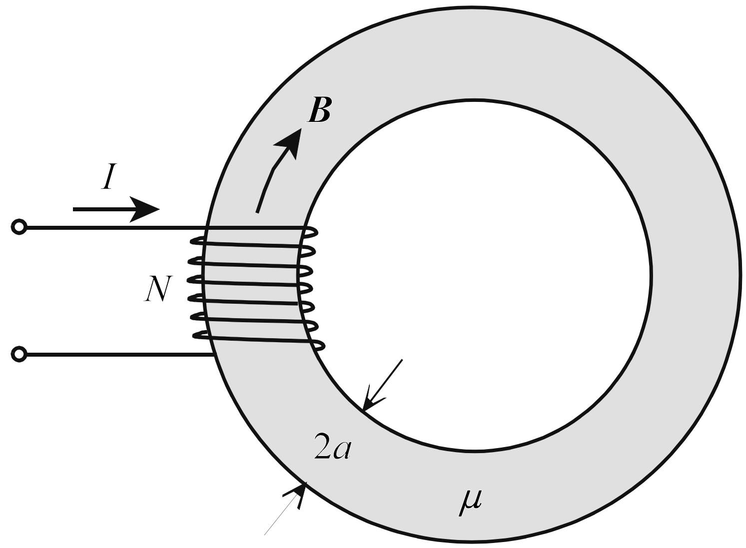
If the long solenoid is bent into the shape of a circle, a *toroid* is formed. The toroid [mean length = $l = 2\pi\rho_o$ (ρ_o = toroid mean radius), core cross-sectional area = A, core permeability = μ, number of turns = N, and current = I] has the advantage that the magnetic field is contained totally within magnetic core, unlike the solenoid. The magnetic circuit for the toroid is identical to that of the solenoid. The only difference is that the length l used in the reluctance calculation is the mean length of the toroid ($l = 2\pi\rho_o$). Inserting the toroid mean length into the solenoid equations for ψ_m and $\boldsymbol{B}$ gives



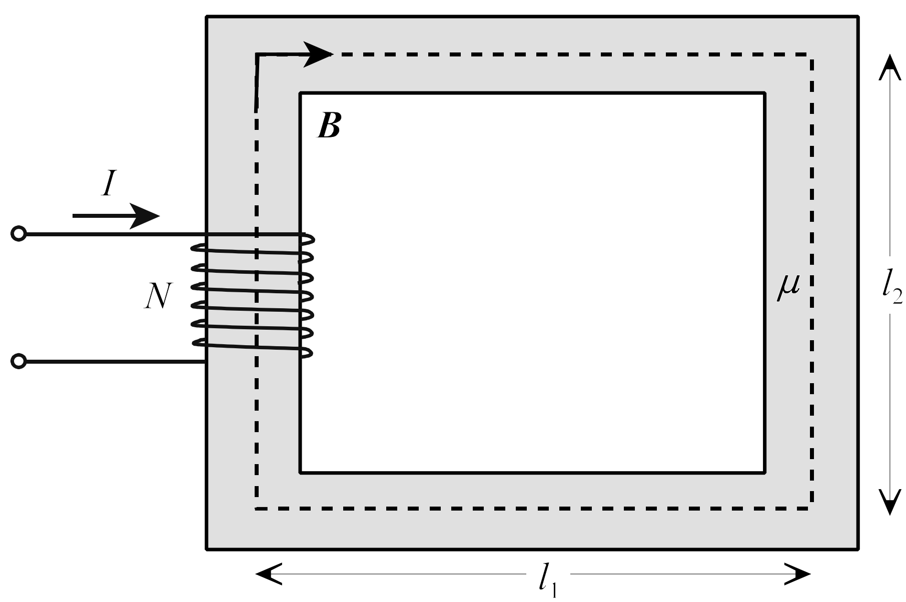
$$\psi_m = \frac{\mu N I A}{l} = \frac{\mu N I A}{2\pi\rho_o}$$

$$B = \frac{\psi_m}{A} = \frac{\mu N I}{2\pi\rho_o} \qquad \text{(toroid)}$$

If the core of the toroid is a ferromagnetic material with a large relative permeability, the turns of the current coil can be placed at any location on the toroid and the previous equation is still valid. This is true since the magnetic reluctance of a ferromagnetic core with a large μ_r is so much smaller than that of the surrounding air, that all of the magnetic flux tends to stay within the core, irregardless of how the coils are arranged.



The magnetic circuit analysis technique can also be used to determine the magnetic flux density in cores of other shapes (rectangular, for instance). Again, the magnetic circuit is identical to that of the toroid. The length l for this geometry is the mean length around the rectangular core [$L_{core} = 2(l_1+l_2)$].



The total magnetic flux and magnetic flux density inside the rectangular core are

$$\psi_m = \frac{\mu NIA}{l} = \frac{\mu NIA}{2(l_1 + l_2)}$$

$$B = \frac{\psi_m}{A} = \frac{\mu NI}{2(l_1 + l_2)} \qquad (\text{rectangular core})$$

As previously shown, the magnetic circuit technique can be applied to more complicated (nonuniform) core geometries. However, in nonuniform cores (multiple flux paths, irregular cross-sections, etc.), multiple equations are necessary to define different values of total magnetic flux and magnetic flux density at different locations in the core.

The total magnetic flux equation for the general uniform core device is useful in determining the self-inductance.

$$\psi_m = \frac{NI}{\Re_{core}}$$

The self-inductance is defined as the ratio of magnetic flux linkage (Λ) to current (I).

$$L \equiv \frac{\Lambda}{I}$$

For any uniform core device, the flux linkage is equal to the number of turns (N) times the total magnetic flux (ψ_m). The inductance of the uniform core device becomes

$$L \equiv \frac{N\psi_m}{I} = \frac{N\left(\dfrac{NI}{\Re_{core}}\right)}{I} = \frac{N^2}{\Re_{core}} = \frac{\mu N^2 A}{l}$$

Applying the inductance equation to the three previous uniform core examples gives

$$L = \frac{\mu N^2 \pi a^2}{l} \qquad \text{(long solenoid)}$$

$$L = \frac{\mu N^2 A}{2\pi \rho_o} \qquad \text{(toroid)}$$

$$L = \frac{\mu N^2 A}{2(l_1 + l_2)} \qquad \text{(rectangular core)}$$

The determination of the self-inductance for nonuniform core devices, such as magnetic circuits with air gaps, follows according to the equation

$$\psi_m = \frac{NI}{\Re_{total}}$$

where $\Re_{total}$ is the total reluctance seen by the mmf in the equivalent magnetic circuit. Inserting this equation into the self-inductance equation yields

$$L \equiv \frac{N\psi_m}{I} = \frac{N\left(\dfrac{NI}{\Re_{total}}\right)}{I} = \frac{N^2}{\Re_{total}} \qquad \text{(nonuniform core)}$$

LOSSES IN MAGNETIC CIRCUITS

There are two major sources for energy loss in magnetic circuits. These are:

(1) Hysteresis loss
(2) Eddy current loss

Hysteresis Loss

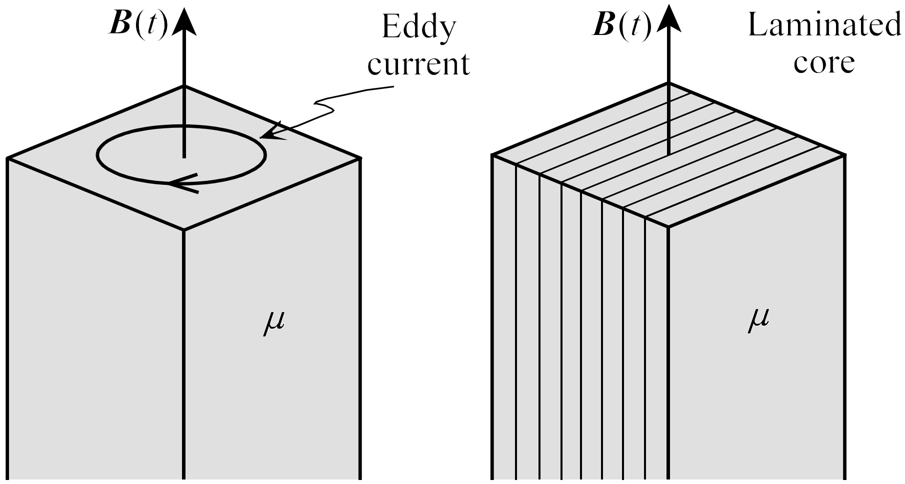
Under AC excitation, the ferromagnetic core undergoes continual hysteresis. There is energy lost in each hysteresis cycle. The energy loss in the ferromagnetic core is in the form of heat caused by the movement of the magnetic dipoles as the excitation field oscillates back and forth. It can be shown that the hysteresis loss (P_h) can be written as

$$P_h = K_h B_{\max}^n f$$

where K_h and n are empirically determined constants dependent on the characteristics of the core material and the core volume. Note that the hysteresis loss varies linearly with the operating frequency.

Eddy Current Loss

Given that the ferromagnetic core in most magnetic circuits is also a a good conductor of current, the time-varying magnetic flux passing through the core can induce circulating currents by Faraday induction. These currents are known as *eddy currents*. Eddy currents can also heat the core due to the ohmic losses in the conductor. The magnitude of the eddy currents can be decreased significantly by using a laminated core in which the laminations are separated by a thin insulating layer (for example, an oxide layer). The eddy currents are reduced since the cross-sectional area available for induction is reduced. Given the orientation of the laminations relative to the core flux density, the reluctance of the core is not significantly diminished if the insulating layers are thin.



The eddy current loss (P_e) in the ferromagnetic core can be written as

$$P_e = K_e B_{\max}^2 f^2$$

where K_e is a constant dependent on the characteristics of the core material and the lamination thickness. Note that the eddy current loss varies as the square of the operating frequency.

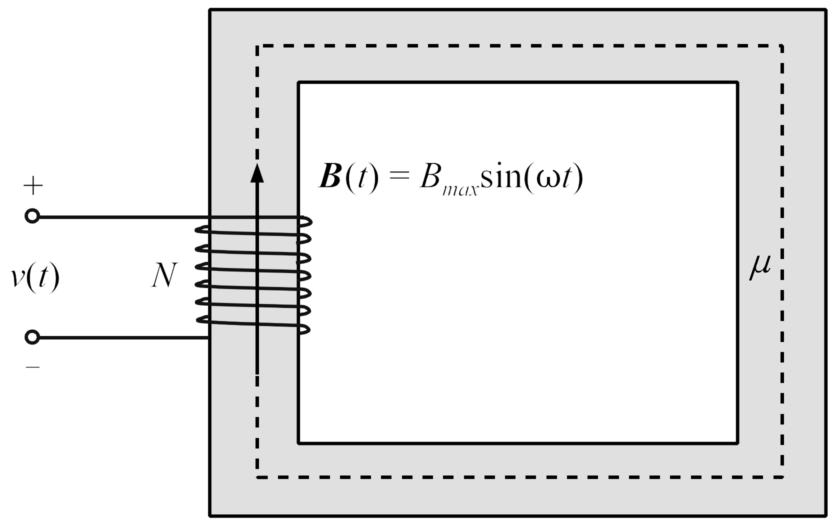
INDUCED VOLTAGES IN COIL-WOUND CORES

Given a sinusoidally-varying magnetic flux density within a coil-wound ferromagnetic core as shown below, a voltage will be induced in the coil. According to Faraday's law, if the magnetic flux density within the core is assumed to be

$$B(t) = B_{\max}\sin(\omega t) = \frac{\psi_m(t)}{A} \qquad (\text{uniform B})$$

then magnitude of the induced voltage is

$$v(t) = N\frac{d\psi_m(t)}{dt} = N\omega B_{\max}A\cos(\omega t) = V_{\max}\cos(\omega t)$$



The peak value of the induced voltage is

$$V_{\max} = N\omega B_{\max}A$$

and the rms value of the induced voltage is

$$V_{rms} = \frac{V_{\max}}{\sqrt{2}} = \frac{N2\pi f B_{\max}A}{\sqrt{2}} = 4.44 N f B_{\max} A$$

This equation gives the relationship between the rms voltage measured on a coil-wound core under sinusoidal excitation.