EEM 308 Introduction to Communications

Lecture 2

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Power and Energy

The energy and power of a signal represent the energy or power delivered by the signal when it is interpreted as a voltage or current source feeding a 1 Ω resistor.
Energy Content of $x(t)$:

$$E_x = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1}$$

Power content of $x(t)$:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2}$$

Practically all periodic signals are power-type and have power

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt \tag{3}$$

- A signal is energy-type if $E_x < \infty$
- A signal is power-type if $0 < P_x < \infty$
- A signal cannot be both power-type and energy-type.
- A signal can be neither energy-type nor power-type

Energy-Type Signals

- For an energy-type signal $x(t)$, we define the autocorrelation function

$$\begin{aligned} R_x(\tau) &= x(\tau) \star x^*(-\tau) \\ &= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \\ &= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt \end{aligned} \tag{4}$$

- By setting $\tau = 0$, we obtain its energy content,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = R_x(0) \tag{5}$$

- According to the autocorrelation theorem,

$$\mathcal{F}\{R_x(\tau)\} = |X(f)|^2 = \text{energy spectral density} = G_x(f) \tag{6}$$

- Using Rayleigh's theorem we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{7}$$

- Energy content of $x(t)$ is also equal to the integral of the energy spectral density over all frequencies,

$$E_x = \int_{-\infty}^{\infty} G_x(f)df \tag{8}$$

Example: Energy-type signals

Determine the autocorrelation function, energy spectral density, and the energy content of the signal $x(t) = e^{-\alpha t} u_{-1}(t)$, $\alpha > 0$.

EXAMPLE

SOLUTION

Power-Type Signals

The time-avg autocorrelation function of the power-type signal

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

If $\tau = 0$, $R_x(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = P_x$

Power spectral density of the signal $x(t)$:

$$S_x(f) = \mathcal{F}\{R_x(\tau)\}$$

$$P_x = \int_{-\infty}^{\infty} S_x(f)df$$

Review: Linear and Time-invariant (LTI) Systems

- A system is **linear** if a linear combination of the inputs result in the corresponding linear combination of outputs

$$\begin{matrix} x_1(t) \to y_1(t) \\ x_2(t) \to y_2(t) \end{matrix} \quad \Rightarrow \quad ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

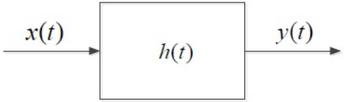
- A system is **time-invariant** if a time-shift of its input results in corresponding time-shift of its output, i.e. the system does not change with time

$$x(t) \to y(t) \quad \Rightarrow \quad x(t-\tau) = y(t-\tau)$$

- A system is called **LTI** if it satisfies the linearity and time-invariance property

$$\begin{matrix} x_1(t) \to y_1(t) \\ x_2(t) \to y_2(t) \end{matrix} \quad \Rightarrow \quad ax_1(t-\tau) + bx_2(t-\tau) \to ay_1(t-\tau) + by_2(t-\tau)$$

Passing Power-Type Signals through LTI Systems



The output

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

The time-avg autocorrelation function for the output

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} y(t)y^*(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \overbrace{\int_{-\infty}^{\infty} h(u)x(t-u)du}^{y(t)} \overbrace{\int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)dv}^{y^*(t-\tau)}\, dt$$

$$= R_x(\tau) * h(\tau) * h^*(-\tau)$$

Taking the FT of both sides:

$$S_y(f) = S_x(f)H(f)H^*(f) = S_x(f)|H(f)|^2$$

POWER-TYPE SIGNALS:
PERIODIC SIGNALS

All nonzero periodic signals are power-type!

Let $x(t)$ be a periodic signal with period T_0

$$\begin{aligned} R_x(\tau) &= \lim_{T\to\infty} \tfrac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt \\ &= \vdots \\ &= \tfrac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt \quad \text{finite integral} \end{aligned}$$

Substituting the FS expansion $R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi \frac{n}{T_0}\tau}$

For periodic signals, $R_x(\tau)$ is periodic with period T_0

$$S_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$

$$P_x = \int_{-\infty}^{\infty} S_x(f)df = \sum_{n=-\infty}^{\infty} |x_n|^2$$

If a periodic signal passes through an LTI system with frequency response $H(f)$, the output will be periodic.
The power spectral density of the output:

$$\begin{aligned} S_y(f) &= |H(f)|^2 S_x(f) \\ &= |H(f)|^2 \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \tfrac{n}{T_0}) \\ &= \sum_{n=-\infty}^{\infty} |x_n|^2 \left| H\left(\tfrac{n}{T_0}\right) \right|^2 \delta(f - \tfrac{n}{T_0}) \end{aligned}$$

The power content of the output:

$$P_y = \sum_{n=-\infty}^{\infty} |x_n|^2 \left| H\left(\frac{n}{T_0}\right) \right|^2$$

Example: Power Type Signals

Determine the power contents of the signal $x_1(t) = A\cos(2\pi f_0 t + \theta)$, and signal $x_2(t) = Au_{-1}(t)$.

EXAMPLE:

SOLUTION

EXAMPLE:

Classify the signal $x(t)$ into energy-type signal, power-type signal and signal that is neither energy-type nor power-type signal.

Hilbert Transform

- Does not involve a change of domain. Signals are completely different.
- In Fourier, Laplace, and z-transforms, the resulting two signals are equivalent representations of the same signal in terms of two different arguments, time and frequency.
- The Hilbert transform of a signal $x(t)$ is a signal $\hat{x}(t)$ whose frequency components lag the frequency components of $x(t)$ by 90°

$$\hat{x}(t) = x(t) \star \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \tag{9}$$

$$\mathcal{F}[\hat{x}(t)] = \hat{X}(f) = -j\text{sgn}(f) X(f) \tag{10}$$

since $\frac{1}{\pi t} \iff -j\text{sgn}(f)$.
Amplitude (so the energy and the power) is not affected, only the phase

EXAMPLE:

Determine the Hilbert transform of the signal $x(t) = cos 2\pi f_c t$.

EXAMPLE

Hilbert Transform

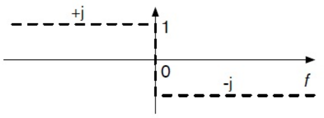
An import filter has an impulse response

$$h(t) = \frac{1}{\pi t}$$

By duality, using the $\mathrm{sgn}(t)$ transform we found above,

$$H(f) = -j\ \mathrm{sgn}(f)$$

which looks like this

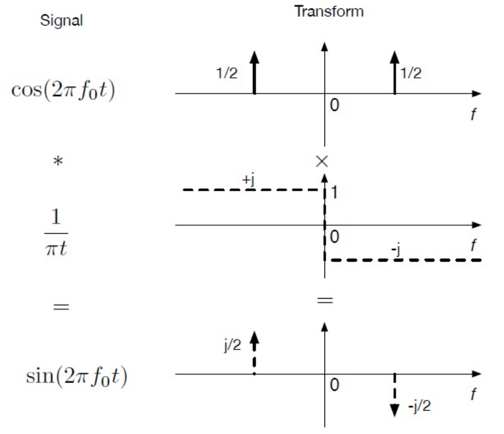


To see why this is important, consider

$$\cos(2\pi f_0 t) * \frac{1}{\pi t}$$

What does this do?

Hilbert Transform



It has turned a cosine into a sine! This will turn up frequently.

EXAMPLE

Determine the Hilbert transform of the signal $x(t) = 2sinc(2t)$.

HT Properties

- **Evennes and Oddness.** The Hilbert transform of an even signal is odd, and the Hilbert transform of an odd signal is even.
- **Sign Reversal.** Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal,
$$\hat{\hat{x}}(t) = -x(t)$$
- **Energy.** The energy content of a signal is equal to the energy content of its Hilbert transform.
- **Orthogonality.** The signal $x(t)$ and its Hilbert transform are orthogonal,
$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0 \tag{11}$$

Lowpass and Bandpass Signals

- A *lowpass* signal is a signal in which the spectrum (frequency content) of the signal is located around the zero frequency.
- A *bandpass* signal is a signal with a spectrum far from the zero frequency.
- The frequency spectrum of a bandpass signal is usually located around a frequency f_c, which is much higher than the bandwidth of the signal
  - (recall that the bandwidth of a signal is the set of the range of all positive frequencies present in the signal).

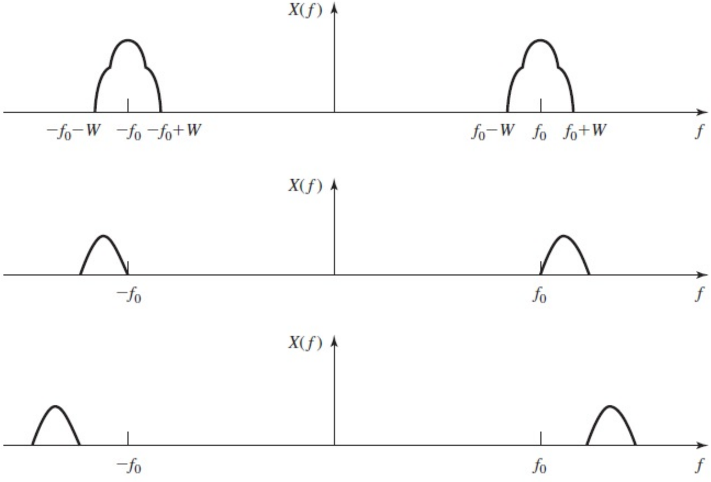


Figure. Examples of bandpass signals

PRE-ENVELOPE

- An analytic signal $x_p(t)$ or $x_+(t)$, corresponding to the real signal $x(t)$, is defined as

$$x_p(t) = x(t) + j\hat{x}(t) \tag{12}$$

where $\hat{x}(t)$ is the Hilbert transform of $x(t)$.
- The *envelope* of a signal is defined mathematically as the magnitude of the analytic signal $x_p(t)$.
- The spectrum of the analytic signal is also of interest.
- Fourier transform of $x_p(t)$ is,

$$X_+(f) = X_p(f) = X(f) + j\{-jsgn(f)X(f)\} \tag{13}$$

- The result is,

$$X_p(f) = X(f)[1 + sgn(f)] \tag{14}$$

- or

$$X_p(f) = \begin{cases} 2X(f), & f > 0 \\ 0, & f < 0 \end{cases} \tag{15}$$

- or

$$X_p(f) = 2u(f)X(f) \tag{16}$$

Pre-Envelope

Proof

Complex Envelope

- $x_p(t)$, pre-envelope of $x(t)$, can be written as,

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} \tag{17}$$

  where $\tilde{x}(t)$ is a complex-valued lowpass representation of bandpass signal (complex envelope).
- $\tilde{x}(t)$ can be first found as,

$$\tilde{x}(t) = x_p(t)e^{-j2\pi f_0 t} \tag{18}$$

- Second, we can find $\tilde{x}(t)$ by using a frequency-domain approach to obtain $X(f)$, then scale its positive frequency components by a factor of 2 to give $X_p(f)$, and translate the resultant spectrum by f_0 Hz to the left. The inverse Fourier transform of this translated spectrum is then $\tilde{x}(t)$.

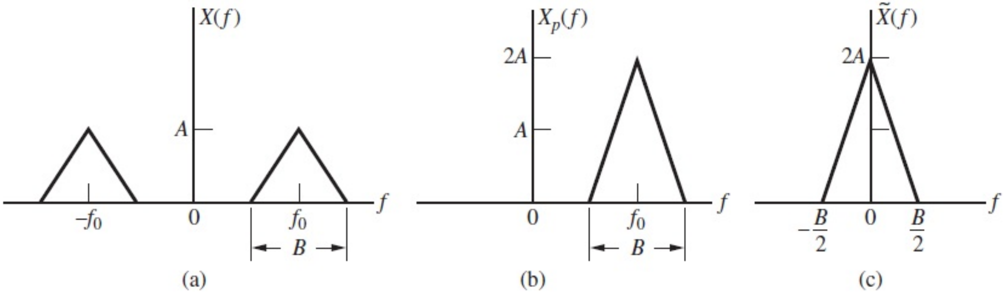


Figure. Spectra pertaining to the formation of a complex envelope of a signal $x(t)$. (a) A bandpass signal spectrum. (b) Twice the positive-frequency portion of $X(f)$ corresponding to $\mathcal{F}[x(t) + j\hat{x}(t)]$ (c) Spectrum of $\tilde{x}(t)$

In-phase and Quadrature Components

In general, $\tilde{x}(t)$ is a complex signal.

Let $\tilde{x}(t) = x_c(t) + jx_s(t)$

$x_c(t)$: in-phase

$x_s(t)$: quadrature components of the bandpass signal $x(t)$.

$$\begin{aligned}
x_p(t) &= \tilde{x}(t)e^{j2\pi f_0 t} \\
&= [x_c(t) + jx_s(t)]\, e^{j2\pi f_0 t} \\
&= [x_c(t) + jx_s(t)]\, [\cos(2\pi f_0 t) + j\sin(2\pi f_0 t)] \\
&= [x_c(t)\cos(2\pi f_0 t) - x_s(t)\sin(2\pi f_0 t)] \\
&\cdots + j\, [x_c(t)\sin(2\pi f_0 t) + x_s(t)\cos(2\pi f_0 t)]
\end{aligned}$$

Recall $x_p(t) = x(t) + j\hat{x}(t)$. Thus,

$$\begin{aligned}
x(t) &= x_c(t)\cos(2\pi f_0 t) - x_s(t)\sin(2\pi f_0 t) \\
\hat{x}(t) &= x_c(t)\sin(2\pi f_0 t) + x_s(t)\cos(2\pi f_0 t)
\end{aligned}$$

a bandpass signal can be represented in terms of two lowpass signals, namely, its in-phase and quadrature components.

EXAMPLE

Consider the real bandpass signal $x(t) = \cos(22\pi t)$. Find the pre-envelope, complex envelope, in-phase and quadrature components of $x(t)$. ($f_0 = 10$ Hz)

Envelope (Real Envelope)

$$\tilde{x}(t) = x_c(t) + jx_s(t) \tag{19}$$

$$a(t) = |\tilde{x}(t)| = \sqrt{x_c^2(t) + jx_s^2(t)} = |x_p(t)| \tag{20}$$

$$\phi(t) = \tan^{-1}\left(\frac{x_s(t)}{x_c(t)}\right) \tag{21}$$

Complex envelope $\tilde{x}(t)$ in polar form:

$$\tilde{x}(t) = a(t)e^{j\phi(t)} \tag{22}$$

Pre-envelope:

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t} = a(t)e^{j\phi(t)}e^{j2\pi f_0 t} = x(t) + j\hat{x}(t) \tag{23}$$

$$x(t) = a(t)\cos(2\pi f_0 t + \phi(t)) \tag{24}$$

EXAMPLE:

Consider an RF pulse

$$x(t) = Arect\left(\frac{t}{T}\right)\cos(2\pi f_c t)$$

where $\frac{1}{T} << f_c$. Find the pre-envelope, complex envelope and real envelope.

Example:

Solution

Example:

Solution

TABLE 2.1 TABLE OF FOURIER-TRANSFORM PAIRS

Time Domain	Frequency Domain
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$
$\Pi(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi\,\mathrm{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$