1. Consider the linear time-invariant system whose block diagram is shown below. Determine all values of $K_1$ and $K_2$ for which the closed-loop system is asymptotically stable (assume that each block is minimal). Indicate those values on the $K_1$ vs. $K_2$ plane.

![Block Diagram](image)

2. Consider the linear time-invariant systems whose transfer function matrices are given below. For each system, (i) determine whether or not the given system is bounded-input bounded-output stable; (ii) if the system is unstable, find a feedback controller to stabilize the system (find the transfer function matrix of the controller and draw a block diagram of the closed-loop system).

   a) $G(s) = \frac{s^2 + 7s + 12}{s^3 + 3s^2 + 4s + 2}$
   b) $G(s) = \frac{s + 2}{s^3 - s}$
   c) $G(s) = \begin{bmatrix} \frac{s + 2}{s - 1} \\ 2 \\ \frac{2}{s^2 - 1} \end{bmatrix}$

3. Consider the system given in part (a) of Question 2. (i) Determine whether or not all the poles of the system have real parts less than $-2$. (ii) If the system has a pole with real part greater than or equal to $-2$, find a feedback controller so that all the closed loop poles have real parts less than $-2$ (find the transfer function matrix of the controller and draw a block diagram of the closed-loop system).

4. Consider the system described by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -38 & -28 & -9 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Find the right-most eigenvalue(s) of the dynamics matrix without finding all the eigenvalues. Is the system stable? Why?
EEU342: Fundamentals of Control Systems

Solutions of Homework #7

1. The transfer function of the connection of the inner feedback is:

\[ G_1(s) = \frac{s+2}{s^2 - 1} \]

\[ = \frac{s+2}{1 + K_1 \cdot \frac{s+2}{s^2 - 1}} \]

and the transfer function of the overall system is:

\[ G_{cl}(s) = \frac{G_1(s) \cdot \frac{1}{s}}{1 + K_2 \cdot G_1(s) \cdot \frac{1}{s}} \]

\[ = \frac{s+2}{s(s^2 - 1 + K_1(s+2)) + K_2(s+2)} \]

Therefore, the characteristic equation of the closed-loop system is:

\[ d(s) = s^3 + K_1 s^2 + (2K_1 + K_2 - 1) s + 2K_2 \]

We can apply Routh's Hurwitz test to \( d(s) \), to determine all values of \( K_1 \) and \( K_2 \) for which all poles of closed-loop system will be inside left half plane (LHP).

\[
\begin{array}{c|cccc}
 s^3 & 1 & 2K_1 + K_2 - 1 & \underline{K_1 > 0, K_2 > 0} \\
 s^2 & K_1 & 2K_2 & \text{if } K_1 < 2, \quad \underline{K_2 < (2K_1^2 - K_1) / (2 - K_1)} \\
 s^1 & 2K_1^2 + K_1K_2 - K_1 - 2K_2 & \text{if } K_1 > 2, \quad \underline{K_2 > (2K_1^2 - K_1) / (2 - K_1)} \\
 s^0 & 2K_2 & \text{} & \text{} \\
\end{array}
\]

\[ 2K_1^2 + K_1K_2 - K_1 - 2K_2 > 0 \]
\[ 2K_1^2 - K_1 > 2K_2 - K_1K_2 \]
\[ 2K_1^2 - K_1 > K_2(2 - K_1) \]
2. a) \( G(s) = \frac{s^2 + 7s + 12}{s^3 + 3s^2 + 4s + 2} \)

Routh's Hurwitz test:

\[
\begin{array}{c|ccc}
S^3 & 1 & 4 & \text{Since there is no sign change, the system is BIBO stable.} \\
S^2 & 3 & 2 & \\
S^1 & 10/3 & \\
S^0 & 2 & \\
\end{array}
\]

b) \( G(s) = \frac{s + 2}{s^3 - 5} \)

i. Routh's Hurwitz test:

\[
\begin{array}{c|ccc}
S^3 & 1 & -1 & \text{one sign change \Rightarrow one pole in open right half-plane (ORHP)} \\
S^2 & 0 & 0 & \\
S^1 & -1 & \\
S^0 & 0 & \\
\end{array}
\]

one pole at zero.

Thus, the system is not BIBO stable.
ii. If we let $C(s) = K$, then closed loop system become:

\[
\begin{align*}
+ & \quad \rightarrow \quad G(s) \quad \rightarrow \quad - \quad \rightarrow \quad C(s) \\
\end{align*}
\]

the transfer function matrix of cl. system is:

\[
G_{cl}(s) = \frac{G(s)}{1 + C(s)G(s)}
\]

\[
= \frac{\frac{s+2}{s^3-s}}{1 + K \frac{s+2}{s^3-s}} = \frac{s+2}{(s^3-s) + K(s+2)}
\]

the characteristic equation of $G_{cl}(s)$ is:

\[
d(s) = (s^3-s) + K(s+2)
\]

\[
= s^3 + (K-1)s + 2K
\]

Routh’s Hurwitz test:

\[
\begin{array}{cccc}
 s^3 & 1 & K-1 \\
 s^2 & 0 & 2K \\
 s^1 & K-1-2K & e \\
 s^0 & 2K \\
\end{array}
\]

there will be either a sign change (a pole in ORHP) or a situation of same sign below and above $e$ (a pair of imaginary poles)

Thus, the system can not be stable with the controller $C(s) = K$. 
Thus we need a first-order controller of the form \( C(s) = \frac{k_1 s + k_2}{s + p} \).

Let's choose \( p = 5 \), then the characteristic equation becomes:

\[
d(s) = (s^3 - 5)(s + 5) + (k_1 s + k_2)(s + 2)
\]

\[
= s^4 + 5s^3 + (k_1 - 1)s^2 + (2k_1 + k_2 - 5)s + 2k_2
\]

Routh's Hurwitz test:

| \( s^4 \) | 1 & \( k_1 - 1 \) & 2k_2 & K_2 > 0 & \( 3k_1 - 2k_2 > 0 \), \( k_1 > \frac{k_2}{3} \) |
| \( s^3 \) | 5 & 2k_1 + k_2 - 5 |
| \( s^2 \) | \( \frac{3k_1 - 2k_2}{5} \) & 2k_2 |
| \( s^1 \) | \( 6k_1^2 - k_2^2 + 15k_1 - 45k_2 \) |
| \( s^0 \) | 2k_2 |

Let's choose \( k_1 = 6 \), \( k_2 = 2 \). The controller is:

\( C(s) = \frac{6s + 2}{s + 5} \)

C) i. \( G(s) = \begin{bmatrix} \frac{s + 2}{s - 1} \\ \frac{s}{s^2 + 1} \end{bmatrix} \)

\[
\begin{array}{c|c|c|c|c|c}
   & 1 & -1 & s & 1 \\

e = 0 & s^2 & s^1 & s^2 & 1 \\
-1 & s^0 & -1 & s^0 & -1 \\
\end{array}
\]

one sign change \( \Rightarrow \) one pole at ORHP

Thus, the system is not stable!
When we design a controller, the block diagram of the closed-loop system will be:

![Block Diagram](image)

and the transfer function of the closed-loop system:

$$G_c(s) = G(s) C(s) \left( I + G(s) C(s) \right)^{-1}$$

$$= G(s) \left( I + G(s) C(s) \right)^{-1} C(s)$$

$$= \frac{1}{1 + C(s) G(s)} G(s) C(s)$$

$$= \frac{1}{d(s)} N(s)$$

If $C(s) = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, the characteristic equation $d(s)$ will be:

$$d(s) = K_1(s+2)(s+1) + 2K_2 + (s^2-1)$$

$$= (K_1+1)s^2 + 3K_1 s + 2K_1 + 2K_2 - 1$$

Routh's Hurwitz test:

<table>
<thead>
<tr>
<th>s^2</th>
<th>K_1+1</th>
<th>2K_1+2K_2-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^1</td>
<td>3K_1</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>2K_1+2K_2-1</td>
<td></td>
</tr>
</tbody>
</table>

In order to obtain no sign change:

- If $K_1+1 > 0$, $K_1 > -1$
- $2K_1+2K_2-1 > 0$, $K_2 > \frac{1}{2} - K_1$
- If $K_1+1 < 0$, $K_1 < -1$
- $2K_1+2K_2-1 < 0$, $K_2 < \frac{1}{2} - K_1$

Let’s choose $K_1 = K_2 = 1$. The controller is:

$$C(s) = \begin{bmatrix} 1 & 1 \end{bmatrix}$$
3. i. \( G(s) = \frac{s^2 + 7s + 12}{s^2 + 3s^2 + 4s + 2} = \frac{n(s)}{d(s)} \)

If we apply Routh Hurwitz test to \( d(s + \sigma) \) instead of \( d(s) \). This gives us information about the poles in the region of right side of \( s = \sigma \).

Thus, in this question, we should apply the Routh’s Hurwitz test to \( d(s-2) \)

\[
d(s-2) = (s-2)^3 + 3(s-2)^2 + 4(s-2) + 2
\]

\[
= s^3 - 3s^2 + 4s - 2
\]

\[
\begin{array}{c|ccc}
 s^3 & 1 & 4 & \text{3 sign changes} \\
 s^2 & -3 & -2 & \Rightarrow \text{3 poles with} \\
 s^1 & 10/3 & \text{real part greater} \\
 s^0 & -2 & \text{than -2}. \\
\end{array}
\]

Let's design a feedback controller in the form of \( C(s) = K \).

In this case, the characteristic equation of closed-loop system is:

\[
d(s) = s^3 + 3s^2 + 4s + 2 + K(s^2 + 7s + 12)
\]

\[
= s^3 + (3 + K)s^2 + (4 + 7K)s + 2 + 12K
\]

\[
d(s-2) = (s-2)^3 + (3 + K)(s-2)^2 + (4 + 7K)(s-2) + 2 + 12K
\]

\[
= s^3 + (K-3)s^2 + (3K + 4)s + (2K - 2)
\]
\[ s^3 + 3s^2 + 3k + 4 \quad k - 3 > 0 \implies k > 3 \\
\]
\[ s^2 + k - 3 + 2k - 2 \quad 2k - 2 > 0 \implies k > 1 \\
\]
\[ s^1 \frac{(k-3)(3k+4)-(2k-2)}{k-3} \quad (k-3)(3k+4)-(2k-2) > 0 \\
\]
\[ s^0 + 2k - 2 \quad 3k^2 - 7k - 10 > 0 \quad k < -1 \lor k > 3.33 \]

Thus we can choose a controller \( C(s) = k \quad (k > 3.33) \) in order to replace all closed-loop poles at the left side of -2.

\[
\begin{array}{c}
\text{r} \\
\text{o} \\
\text{G(s)} \\
\text{C(s)} \\
\text{y} \\
\end{array}
\]

4. \( s^4 \dot{x}(t) = Ax(t) + Bu(t) \)

the characteristic equation of the system is:
\[
Q(s) = \det(sI - A) = s^4 + 9s^3 + 28s^2 + 38s + 24
\]

we can find the right-most eigenvalues of the system by applying the Routh's Hurwitz test to the equation \( d(s - \sigma) \).
\[ d(s-\sigma) = (s-\sigma)^4 + 9(s-\sigma)^3 + 28(s-\sigma)^2 + 38(s-\sigma) + 24 \]

\[ = s^4 + (-4s + 9)s^3 + (6s^2 - 27s + 28)s^2 + \]

\[ + (-4s^3 + 27s^2 - 56s + 38)s + \]

\[ + (s^4 - 9s^3 + 28s^2 - 38s + 24) \]

\[ \equiv A \]

\[ S^4 \equiv B \]

\[ S^3 \equiv C \]

\[ S^2 \equiv D \]

\[ \frac{AB - CE}{E} \equiv D \]

\[ \frac{EC - AD}{E} \equiv D \]

\[ A > 0 \Rightarrow -4s + 9 > 0 \Rightarrow \sigma < 2.25 \]

\[ E > 0 \Rightarrow AB - CE > 0 \Rightarrow -20s^3 + 135s^2 - 299s + 214 > 0 \Rightarrow \sigma < 1.56 \]

\[ \frac{EC - AD}{E} \Rightarrow 6s^6 - 864s^5 + 47824s^4 - 138960s^3 + 222640s^2 - 18540s + 6188 > 0 \]

\[ \Rightarrow \sigma < 1 \lor \sigma > 7.5 \]

\[ D > 0 \Rightarrow \sigma^4 - 9\sigma^3 + 28\sigma^2 - 38\sigma + 24 > 0 \]

\[ \Rightarrow \sigma < 3 \lor \sigma > 4 \]

The smallest upper bound for \( \sigma \) is 1. We found this result from \( d(s-\sigma) \). So, the real part of the rightmost eigenvalue(s) of the system is -1. It means, all eigenvalues of system are inside the LHP, the system is stable.
To find the imaginary parts of the eigenvalues with real part = -1, we write the auxiliary polynomial from the $s^2$ row (the row above the $s$ row) with $\sigma = 1$:

$$E \bigg|_{\sigma = 1} s^2 + D \bigg|_{s = 1} = 6s^2 + 6 = 0 \Rightarrow s^2 + 1 = 0$$

writing $s = j\omega \Rightarrow -\omega^2 + 1 = 0 \Rightarrow \omega = \pm 1$

$\Rightarrow$ Right-most eigenvalues are: $-1 \pm j$