1. Consider the linear time-invariant (LTI) system which has the transfer function \( G(s) = \frac{s + 2}{s^3 + 2s^2 + 2s + 1} \).

   a) Find the real part of the right-most pole(s) of the system without finding all of its poles. Is/are the right-most pole(s) real or complex? How can you tell? Is the system bounded-input bounded-output stable? Why?

   b) Are all the poles of the system have real parts less than -2? Why? If not, design a LTI proper feedback controller so that all the closed-loop poles have real parts less than -2 (find the transfer function of the controller and draw a block diagram of the closed-loop system).

2. Consider a LTI plant which has the transfer function \( G(s) = \frac{s}{s^2 - 1} \). Is it possible to design a LTI proper feedback controller for this system such that (i) the closed-loop system is stable; (ii) steady-state error in response to a step reference is zero; and (iii) steady-state error in response to a ramp reference is no more than 10% of the ramp slope in absolute value?

   If your answer is negative, explain your reason.

   If your answer is positive, design such a controller (find the controller transfer function and draw a block diagram of the closed-loop system). Then, using MATLAB, plot the output and the error signals in response to a unit step and a unit ramp reference signals. Determine the steady-state errors from the error plots. Are the requirements (i)–(iii) satisfied?

3. Repeat Question 2 for \( G(s) = \frac{1}{s^2 - 1} \).

4. Consider the LTI systems with transfer functions:

   a) \( \frac{4}{s^2 + 4s + 4} \)  
   b) \( \frac{s + 4}{s^2 + 4s + 4} \)  
   c) \( \frac{-s + 4}{s^2 + 4s + 4} \)  
   d) \( \frac{4}{s^2 + 2s + 4} \)  
   e) \( \frac{s + 4}{s^2 + 2s + 4} \)  
   f) \( \frac{-s + 4}{s^2 + 2s + 4} \)  
   g) \( \frac{4}{s^2 + 2s + 4} \)  
   h) \( \frac{s + 4}{s^2 + 0.2s + 4} \)  
   j) \( \frac{-s + 4}{s^2 + 0.2s + 4} \)

   For each case, (i) find the poles and the zeros and indicate them on the complex plane; (ii) using MATLAB, obtain a plot of the unit step response of the system; (iii) from the plots, determine the percent overshoot and the peak time (see your text book or “Notes on Transient Response of LTI systems” for the definitions of “percent overshoot” and “peak time”). What are the qualitative differences between the responses in relation with pole-zero locations?

5. Consider the LTI system which has the transfer function \( G(s) = \frac{a\omega_n^2}{(s + a)(s^2 + 2\zeta\omega_ns + \omega_n^2)} \).

   a) Let \( a = 2 \), \( \omega_n = 1 \), and \( \zeta = \frac{1}{4} \). Show the location of the poles of \( G(s) \) on the complex plane. Then, using MATLAB, plot the unit step response of the system. Also plot the unit step responses of the systems with following transfer functions:

   \( G_1(s) = \frac{a}{s + a} \) and \( G_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2} \)

   Compare the three responses obtained. Is the response for the system with transfer function \( G(s) \) close to any one of the responses of the systems with transfer function \( G_1(s) \) and \( G_2(s) \)? If yes, to which one? Why?

   b) Repeat Part (a) for \( a = \frac{1}{4} \), \( \omega_n = \frac{\sqrt{79}}{4} \), and \( \zeta = \frac{8}{\sqrt{79}} \).
1. a) \( G(s) = \frac{s+2}{s^3+2s^2+2s + 1} \)

\[ d(s) = s^3 + 2s^2 + 2s + 1 \]

We can find the right-most pole(s) of the system by applying Routh's Hurwitz test to the equation \( d(s-\sigma) \):

\[ d(s-\sigma) = (s-\sigma)^3 + 2(s-\sigma)^2 + 2(s-\sigma) + 1 \]

\[ = s^3 + (2-3\sigma)s^2 + (3\sigma^2 - 4\sigma + 2)s + (-\sigma^3 - 12\sigma^2 - 2\sigma + 1) \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>1</th>
<th>( 3\sigma^2 - 4\sigma + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>2-3\sigma</td>
<td>( -\sigma^3 + 2\sigma^2 - 2\sigma + 1 )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( -8\sigma^3 + 16\sigma^2 - 12\sigma + 3 )</td>
<td>( \frac{2-3\sigma}{2-3\sigma} )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( -\sigma^3 + 2\sigma^2 - 2\sigma + 1 )</td>
<td></td>
</tr>
</tbody>
</table>

\[ 2 - 3\sigma > 0 \Rightarrow \sigma < \frac{2}{3} \]

\[ -8\sigma^3 + 16\sigma^2 - 12\sigma + 3 > 0 \Rightarrow \sigma < 0.5 \]

\[ -\sigma^3 + 2\sigma^2 - 2\sigma + 1 > 0 \Rightarrow \sigma < 1 \]

The smallest upper bound for \( \sigma \) is 0.5. We found this result from \( d(s-\sigma) \). So, the real part of the right-most pole(s) of the system is -0.5. It means, all poles of the system are on the LHP, then the system is bounded-input bounded-output. The right-most poles form a complex conjugate pair (non-real) since the first element of the third row of the Routh table is 0.
Zero for \( T = 0.5 \) with positive elements both above and below it.

b) Since the right-most poles have real parts -0.5, not all the poles of the system have real parts less than -2. Since the given system is of 3rd order, a 2nd order feedback controller can move all the closed-loop poles to the left of the line \(-2+j\omega\).

The block diagram of the closed-loop system is shown in Figure 1.

![Block Diagram](image)

Figure 1

The transfer function of the closed-loop system is
\[ G_{cl}(s) = \frac{G(s)}{1 + G(s)C(s)} \]

\[ C(s) = \frac{K_1s^2 + K_2s + K_3}{s^2 + K_4s + K_5} \]

The characteristic polynomial of the closed-loop transfer function is:
\[ d_c(s) = (s^2 + 2s^2 + 2s + 1)(s^2 + K_4s + K_5) + (s + 2)(K_1s^2 + K_2s + K_3) \]
\[ = s^5 + (2 + K_4)s^4 + (2 + 2K_4 + K_5 + K_1)s^3 + \]
\[ (1 + 2K_4 + 2K_5 + K_2 + 2K_1)s^2 + (K_4 + 2K_5 + K_3 + 2K_2)s + \]
\[ (K_5 + 2K_3) \]

In order to get all closed-loop poles to the left of -2, we can apply the Routh's Hurwitz method to \( d_c(s-2) \):
\[ d_c(s-2) = (s-2)^5 + (2 + K_4)(s-2)^4 + (2 + 2K_4 + K_5 + K_1)(s-2)^3 \]
\[ + (1 + 2K_4 + 2K_5 + K_2 + 2K_1)(s-2)^2 + (K_4 + 2K_5 + K_3 + 2K_2)(s-2) \]
\[ + (K_5 + 2K_3) \]
\[ d_c(s-2) = s^5 + (K_4 - 8)s^4 + (26 - 6K_4 + K_5 + K_1)s^3 + \\
+ (-43 + 14K_4 - 4K_5 - 4K_3 + K_2)s^2 + \\
+ (36 - 15K_4 + 6K_5 + 4K_3 - 2K_2 + K_3)s + \\
+ (-12 + 6K_4 - 3K_5) \]

Routh's Hurwitz test:

\[
\begin{align*}
S^5 & \quad 1 \quad 26 - 6K_4 + K_5 + K_1 & \triangleq A \\
S^4 & \quad K_4 - 8 \quad -43 + 14K_4 - 4K_5 - 4K_3 + K_2 & \triangleq E \\
S^3 & \quad \frac{(K_4 - 8) \times A - B}{K_4 - 8} & \triangleq F \\
S^2 & \quad \frac{(K_4 - 8) \times A - B^2 - (K_4 - 8)^2 \times C - (K_4 - 8) \times D}{(K_4 - 8) \times F - E} & \triangleq G \\
S^1 & \quad FG - ED & \triangleq H \\
S^0 & \quad D
\end{align*}
\]

We need to choose \( K_1, K_2, K_3, K_4 \) and \( K_5 \) in order to satisfy the conditions:

\( K_4 - 8 > 0 \), \( E > 0 \), \( G > 0 \), \( FG - ED > 0 \), and \( D > 0 \).

If we choose \( K_1 = 669 \), \( K_2 = 1652 \), \( K_3 = 4583 \), \( K_4 = 29 \), \( K_5 = -346 \)
the closed-loop poles are located at: \(-3, -4, -4, -5, -5\).

Thus the transfer function of the controller:

\[ C(s) = \frac{669s^2 + 1652s + 4583}{s^3 + 29s + 346} \]
2. It is not possible to find controller for the system 
\[ G(s) = \frac{s}{s^2-1} \] such that (i) - (iii) are satisfied.

Because the system has a zero at the origin, tracking of a step reference cannot be satisfied.

3. The controller: 
\[ C(s) = \frac{k_1s^2 + k_2s + k_3}{s^2 + k_4s + k_5} \]

The block diagram of closed-loop system is shown in Figure 2.

* The condition (i):

The closed loop transfer function is: (Block diagram of the closed-loop system is shown in Figure 2.)

\[ G_c(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} \]

\[ = \frac{\frac{nc(s)}{s^2-1}dc(s)}{1 + \frac{1}{s^2-1} \frac{nc(s)}{dc(s)}} = \frac{nc(s)}{(s^2-1)dc(s) + nc(s)} \]

the characteristic polynomial of the closed-loop system is:

\[ dc(s) = (s^2-1)dc(s) + nc(s) \]

For the stability, all roots of this polynomial must be on OLHP.

* The condition (ii):

The error of the closed-loop system is:

\[ \hat{e}(s) = \frac{1}{1 + G(s)C(s)} \hat{r}(s) \]
The steady-state error goes to zero means that

\[
\lim_{{t \to \infty}} e(t) = 0 \iff \lim_{{s \to 0}} s \cdot \hat{E}(s) = 0
\]

\[
s \cdot \hat{E}(s) = \frac{1}{1 + \frac{1}{s^2 - 1} \cdot \frac{dc(s)}{dc(s)} + \frac{n(s)}{s^2}} \cdot \frac{1}{s} \quad (r(t) = u(t) \Rightarrow \hat{r}(s) = \frac{1}{s^2})
\]

\[
= \frac{(s^2 - 1) \cdot dc(s)}{(s^2 - 1) \cdot dc(s) + nc(s)}
\]

In order to get \(\lim_{{s \to 0}} s \cdot \hat{E}(s) = 0\), \(nc(s)\) must have a pole at \(s = 0\).

Then \(dc(s)\) can be chosen as \(dc(s) = s^2 + K_4 s\)

\[\text{as the characteristic and ample.}\]

* The condition (iii):

\[
\left| \lim_{{s \to 0}} s \cdot \hat{E}(s) \right| \leq 0.1 \quad \text{for the ramp reference} \quad (r(t) = t, \hat{r}(s) = \frac{1}{s^2})
\]

\[
\lim_{{s \to 0}} s \cdot \hat{r}(s) = \lim_{{s \to 0}} s \cdot \hat{E}(s) \cdot \left| \frac{(s^2 - 1) \cdot dc(s)}{(s^2 - 1) \cdot dc(s) + nc(s)} \cdot \frac{1}{s^2} \right| \leq 0.1
\]

\[
\lim_{{s \to 0}} \left| \frac{(s^2 - 1) \cdot s(s + K_4)}{(s^2 - 1) \cdot s(s + K_4) + (K_4 s^2 + K_2 s + K_3)} \right| \leq 0.1
\]

\[
\frac{K_4}{K_3} \leq \frac{1}{10} \Rightarrow K_3 \geq 10K_4
\]

we may choose \(K_3 = 11K_4\)
If we can turn back to condition (i):

\[ dc(s) = (s^2 - 1) dc(s) + nc(s) \]

\[ = (s^2 - 1) (s^2 + K_4 s) + (K_1 s^2 + K_2 s + K_3) \]

\[ = s^4 + K_4 s^3 + (K_1 - 1) s^2 + (K_2 - K_4) s + K_3 \]

The Routh’s Hurwitz test for \( dc(s) \):

\[
\begin{array}{ccccccc}
S^4 & 1 & K_1 - 1 & 11K_4 \\
S^3 & K_4 & K_2 - K_4 & \\
S^2 & \frac{K_1 K_4 - K_2}{K_4} & 11K_4 & \\
S^1 & \frac{K_1 K_2 K_4 - K_1 K_4^2 - K_1 K_2 K_4 - 11 K_4^3}{K_1 K_4 - K_2} & \\
S^0 & 11K_4 & \\
\end{array}
\]

1. \( K_4 > 0 \)
2. \( K_1 K_4 - K_2 > 0 \)
3. \( K_1 K_2 K_4 - K_1 K_4^2 - K_1 K_2 K_4 - 11 K_4^3 > 0 \)
4. \( \ldots \)

We may choose \( K_1 = 8 \), \( K_2 = 4 \), \( K_4 = 1 \), \( K_3 = 11 \). \( K_4 = 11 \)

The closed-loop poles are:

\[-0.3798 + i 2.0128, -0.124 + i 1.6147 \]

\[ C(s) = \frac{8 s^2 + 4 s + 11}{s(s + 1)} \]

The graphics of output and error signals in the case of step and ramp references are shown in Figures 3 and 4.
The output is tracking the step reference.

The steady-state error is zero.

**Figure 3.a**

**Figure 3.b**
Output signal & Ramp Reference Signal

Figure 4.a

Error Signal in the case of Ramp Reference

The steady-state error $\leq 0.1$

Figure 4.b
ii. The unit step responses of the systems (a)-(j) are show in the Figures 5-13, respectively.
Figure 8

Figure 9
Figure 10

Figure 11
Figure 12

Figure 13
iii. Percent overshoot:

(a)-(c): There is no overshoot

(d): $M_p = \frac{y_P - K}{K} = \frac{1.163 - 1}{1} = 0.168$

(e): $M_p = \frac{1.196 - 1}{1} = 0.196$

(f): $M_p = \frac{1.178 - 1}{1} = 0.178$

(g): $M_p = \frac{1.852 - 1}{1} = 0.852$

(h): $M_p = \frac{1.956 - 1}{1} = 0.956$

(i): $M_p = \frac{1.952 - 1}{1} = 0.952$

Peak time:

(a)-(c): There is no overshoot

(d): $T_p = 1.8\text{ s.}$

(e): $T_p = 1.5\text{ s.}$

(f): $T_p = 2\text{ s.}$

(g): $T_p = 1.6\text{ s.}$

(h): $T_p = 1.3\text{ s.}$

(i): $T_p = 1.8\text{ s.}$
* Differences between the responses in relation with pole-zero locations:

First, let us examine the systems with no zero \((\text{parts (a),(c),(f)} \text{ and (g)})\). Since \(z=1\) in part (a) (critically damped case), both poles are located on the real axis at \(-w_0 = -2\). Thus, there is no overshoot in the step response of this system as shown in Figure 5.

In part (d) and (g), since \(z<1\) (underdamped case) poles form a complex conjugate pair. The distance from any one of the pole to the origin is \(w_0 = 2\). Because of these complex conjugate pole pairs, the step responses of these systems have overshoots. Since the damping ratio is smaller in part (g), the step response is more oscillatory (near the instability).

The effects of zeros are seen from the graphics of the other parts. The transient response of the system deviate from the systems with no zero. However, the steady-state response remains unchanged. If there is an ORHP zero \((\text{as in part (c),(f) and (g)})\) the transient response initially tends to a opposite direction to that of the steady-state value.

The location of the zeros also effects the peak time and overshoot. As seen from the graphics, in the case of OLHP zero, the peak time is getting smaller and there is a larger overshoot. On the other hand, in the case of ORHP zero, the peak time is longer than the previous one, and the overshoot is smaller than that of the system with OLHP zero, but it is larger than that of the system with no zero.
5. \( a) \quad G(s) = \frac{2}{(s+2)(s^2+0.5s+1)} \)

\[ G_1(s) = \frac{2}{s+2}, \quad G_2(s) = \frac{1}{s^2+0.5s+1} \]

The step responses of the systems \( G(s) \), \( G_1(s) \) and \( G_2(s) \) are shown in Figure 14. The response of \( G(s) \) is close to that of \( G_2(s) \). Because the poles of \( G_2(s) \) are located so close to imaginary axis compared to that of \( G_1(s) \). Thus, poles of \( G_2(s) \) are dominant poles and then transient response of \( G(s) = G_1(s) \cdot G_2(s) \) is dominated by the response of \( G_2(s) \).

\( b) \quad G(s) = \frac{19.75}{(s+0.25)(16s^2+64s+79)} \)

\[ G_1(s) = \frac{0.25}{s+0.25} \]

\[ G_2(s) = \frac{79}{16s^2+64s+79} \]

The step responses of the systems \( G(s) \), \( G_1(s) \) and \( G_2(s) \) are shown in Figure 15. In this case, the response of \( G(s) \) close to that of \( G_1(s) \). Because the poles of \( G_1(s) \) are dominant in this case.