0. (1 point) Write the group you are registered to on the top-right corner of your answer sheet.

1. Consider the sampled-data system whose block diagram is given below, where the two samplers of period $T$ are synchronous and ZOH indicates a zero order hold.

\[
\begin{array}{c}
\begin{array}{c}
  K \\
  \downarrow \\
  T \\
  \downarrow \\
  ZOH \\
  \downarrow \\
  u \\
  \downarrow \\
  S_1 \\
  \downarrow \\
  y
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
  v \\
  \downarrow \\
  S_2 \\
  \downarrow \\
  w \\
  \downarrow \\
  T
\end{array}
\end{array}
\]

$S_1$ is a continuous-time system and $S_2$ is a discrete-time system (which is synchronous with the samplers) which are defined by the following state equations respectively:

\[
\begin{align*}
S_1: & \quad \dot{x}(t) = x(t) + u(t) \\
    & \quad y(t) = x(t) + u(t) \\
S_2: & \quad z(k + 1) = z(k) + w(k) \\
    & \quad v(k) = z(k) + w(k)
\end{align*}
\]

a) (9 points) Is it possible to choose $K$ (real) and $T$ (real and positive) such that the system is asymptotically stable? Why?

b) (25 points) Find all values of $K$ (real) and $T$ (real and positive) for which this system is stable in the sense of Lyapunov. Indicate those values on the $K$ vs. $T$ plane.

2. (65 points) Consider the linear time-invariant (LTI) continuous-time plants with transfer function

\[
a) \quad G(s) = \frac{s}{s - 1} \quad \text{b) } \quad G(s) = \frac{s + 1}{s - 1}
\]

where the unit of time is seconds. It is known that neither plant has any hidden unstable modes. Each plant may be subject to a constant disturbance. It is desired to design a LTI controller, to be implemented on a digital computer, where the computer output is through a digital-to-analog converter (D/A) which includes a zero order hold and has sampling period $T = 100$ milliseconds. The computer can receive inputs through a number of analog-to-digital converters (A/D) which are all synchronous with the D/A. The closed-loop system must be internally stable and the output of the plant must be able to track constant references with no steady-state error and the steady-state error in response to a ramp reference must not exceed (in absolute value) 10% of the ramp slope. Furthermore, the output should settle to its steady-state value at a rate not slower than that of $e^{-t}$. The output of the D/A can be directly applied to the control input $u$ and both the plant output $y$ and the reference $r$ can be directly applied to the input of one of the A/D’s (i.e., there is no need for any sensors/actuators). For each plant, determine whether it is possible to design such a controller. If not, explain the reason. If so, then either:

First design a continuous-time controller to achieve the objectives. Draw a block diagram for the continuous-time implementation. Then, find the equivalent discrete-time controller. Also write a computer program to implement this controller on the digital computer and draw a block diagram showing the implementation of the actual system.

or:

Directly design a discrete-time controller to achieve the equivalent objectives for the discrete-time equivalent closed-loop system. Also write a computer program to implement this controller on the digital computer and draw a block diagram showing the implementation of the actual system.
1. The discrete-time equivalent of the system is:

\[ G_d(z) = \frac{z-1}{z} \mathcal{Z} \left( \frac{1}{s} \left( 1 + \frac{1}{s-1} \right) \right) = \frac{z-1}{z} \mathcal{Z} \left( \frac{1}{s-1} \right) = \frac{z-1}{z-e^T} \]

and

\[ G_2(z) = 1 + \frac{1}{z-1} = \frac{z}{z-1}. \]

a) Note that an unstable pole-zero cancellation (at \( z = 1 \)) occurs between \( G_d(z) \) and \( G_2(z) \). Therefore, whatever \( K \) and \( T \) are, there always exists a hidden mode on the unit circle (at \( z = 1 \)). Therefore, the system can not be asymptotically stable.

b) Since the only hidden mode is a simple mode on the unit circle, it may be possible to make this system stable in the sense of Lyapunov. For this, \( K \) and \( T \) should be chosen such that all the other modes are either inside the unit circle or are simple modes on the unit circle, except at \( z = 1 \) (since there already is one mode at \( z = 1 \), a second mode at \( z = 1 \) will make this mode multiple). To see whether this is possible, let's look at the locus of the closed-loop poles as \( K \) is varied over real numbers. This is shown in Fig. 1 below, where the hidden mode is not shown. The locus for \( K > 0 \) is shown with a continuous line, where the locus for \( K < 0 \) is shown with a broken line. It is seen that the only non-hidden mode is inside the unit circle when \( K > K_1 \) (this mode is at \( z = 1 \) for \( K = K_1 \), thus \( K = K_1 \) is not allowed) and when \( K < K_2 \) (note that \( K_2 < 0 \), so the condition \(|K| > |K_2| \) translates into \( K < K_2 \); when \( K = K_2 \), the mode is at \( z = -1 \), so \( K = K_2 \) is also allowed for Lyapunov stability). We can find \( K_1 \) and \( K_2 \) by the distance formula:

\[ K_1 = \frac{e^T-1}{1} = e^T - 1 \quad ; \quad |K_2| = \frac{e^T+1}{1} \Rightarrow K_2 = -(e^T+1) \]

Thus, the system is stable in the sense of Lyapunov when either \( K > e^T - 1 \) or \( K \leq -(e^T+1) \). This region is shown in Fig. 2 below. Note that, instead of using the root locus method, we could also explicitly calculate the closed-loop modes or use the modified Routh-Hurwitz test. These methods should also give the same result.

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Fig. 1: Root-locus for the non-hidden mode.

Fig. 2: Region for Lyapunov stability.
2. a) The given plant has a zero at the origin (where the reference and the disturbance has a pole). Therefore, it is not possible to design a controller (continuous-time or discrete-time) to track constant references and/or to reject constant disturbances without violating internal stability.

b) Since the plant does not have any zeros at the origin and does not have any hidden unstable modes, it is possible to design a controller to achieve the stated objectives. To satisfy the tracking and the disturbance rejection objectives, the controller must have a pole at the origin. Let us try a continuous-time controller with a transfer function \( C(s) = \frac{K}{s} \), where \( K \) is a constant gain to be determined. With this controller the poles of the closed-loop system, which is shown in Fig. 3, varies as shown in Fig. 4 for \( K > 0 \). Therefore, by choosing \( K \) large enough, it is possible to collect all the poles to the left of the vertical line passing through \(-1\) (thus satisfy both the stability and the speed of response requirements). Furthermore, by choosing \( K \) large enough, it is also possible to satisfy the steady-state error requirement in response to a ramp reference. With this controller, the closed-loop characteristic polynomial is

\[
p(s) = s^2 + (K - 1)s + K
\]

Let

\[
q(s) = p(s - 1) = s^2 + (K - 3)s + 2
\]

Thus, by applying the Routh-Hurwitz test, all the closed-loop poles are to the left of the vertical line passing through \(-1\) when \( K > 3 \). To satisfy the steady-state error requirement in response to a ramp reference, we must have \( |K_v| \geq 10 \), where

\[
K_v = \lim_{s \to 0} sG(s)C(s) = -K
\]

Thus, we must have \( K \geq 10 \). To satisfy all these requirements, let \( K = 10 \). Thus, \( C(s) = \frac{10}{s} \), which can be realized as

![Fig. 3: Continuous-time implementation.](image)

![Fig. 4: Root-locus for the continuous-time implementation.](image)
\[ \dot{x}(t) = e(t) \]
\[ u(t) = 10x(t) \]

where \( e(t) = r(t) - y(t) \), where \( r \) is the reference, and \( x \) is the state of the continuous-time controller. A discrete-time equivalent for this controller can be described as:

\[ x_d(k + 1) = x_d(k) + Te_d(k) \]
\[ u_d(k) = 10x_d(k) \]

where \( T = 0.1 \), \( e_d(k) = r(kT) - y(kT) \), \( x_d \) is the state of the discrete-time controller, and \( u_d \) is the output of the controller which is applied to the input of the D/A whose output is applied to the control input of the plant. Note that this controller is equivalent to (by defining \( z_d(k) = 10x_d(k) \)):

\[ z_d(k + 1) = z_d(k) + e_d(k) \]
\[ u_d(k) = z_d(k) \]

The following program implements this controller:

```plaintext
z = 0 ; % initial condition
while 0==0 % enter an infinite loop
   u = z ; % compute the controller output
   write u ; % send the computed value of u to D/A
   read r, y ; % read the reference and the plant output from the respective A/D
   e = r - y ; % compute the error
   z = z + e % compute the next state
   wait % wait until the next sampling instant
end
```

Overall implementation of the actual controller is shown in Fig. 5, where \( C_d(z) = \frac{1}{z - 1} \).

![Fig. 5: Actual implementation.](image-url)