0. (1 point) Write the group you are registered to on the top-right corner of your answer sheet.

1. (24 points) Consider the sampled-data system whose block diagram is given below, where the two samplers of period $T$ are synchronous. ZOH indicates a zero order hold and UD indicates a unit delay (a delay element of delay time $T$). Find all values of $K$ (real) and $T$ (real and positive) for which this system is stable. Indicate those values on the $K$ vs. $T$ plane.

2. Consider the system described by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

$$y(t) = \begin{bmatrix} 2 & 4 \end{bmatrix} x(t) + u(t)$$

where $x$, $u$, $d$, and $y$ are, respectively, the state, the control input, the disturbance, and the output of the system, and $t$ is the time variable, which is measured in seconds. It is known that the disturbance, $d(t)$, is a sinusoidal of angular frequency 1 rad/sec with no (i.e., zero) DC component. It is desired to design a LTI controller, to be implemented on a digital computer, where the computer output is through a digital-to-analog converter (D/A) which includes a zero order hold and has sampling period $T = 50$ milliseconds. The computer can receive inputs through a number of analog-to-digital converters (A/D) which are all synchronous with the D/A. The closed-loop system must be internally stable and the output of the plant must be able to track constant references with no steady-state error and the steady-state error in response to a ramp reference must not exceed (in absolute value) 10% of the ramp slope (also note the presence of the disturbance described above). The output of the D/A can be directly applied to the control input $u$ and both the plant output $y$ and the reference $r$ can be directly applied to the input of one of the A/D’s (i.e., there is no need for any sensors/actuators). Determine whether it is possible to design such a controller. If not, explain the reason. If so, then either:

First design a continuous-time controller to achieve the objectives. Draw a block diagram for the continuous-time implementation. Then, find the equivalent discrete-time controller. Also write a computer program to implement this controller on the digital computer and draw a block diagram showing the implementation of the actual system.

or:

Directly design a discrete-time controller to achieve the equivalent objectives for the discrete-time equivalent closed-loop system. Also write a computer program to implement this controller on the digital computer and draw a block diagram showing the implementation of the actual system.

3. Repeat Question 2 for the case when equation (2) is changed to:

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + u(t)$$
1. The discrete-time equivalent of the system is:

\[ G_d(z) = \frac{z-1}{z} Z \left[ \frac{1}{s(s-1)} \right] = \frac{e^T - 1}{z - e^T}. \]

Thus, the closed-loop transfer function is

\[ G_c(s) = \frac{Kz(e^T - 1)}{d(z)}, \]

where \( d(z) = z^2 - e^Tz + K(e^T - 1) \). Then,

\[ p(r) := (r-1)^2d\left(\frac{r+1}{r-1}\right) = \left(1-e^T + K(e^T - 1)\right) r^2 + \left(2-2K(e^T - 1)\right) r + \left(1+e^T + K(e^T - 1)\right) \]

Therefore, for closed-loop stability, we must have either

a) (i) \( 1-e^T + K(e^T - 1) > 0 \), (ii) \( 2-2K(e^T - 1) > 0 \), and (iii) \( 1+e^T + K(e^T - 1) > 0 \);

or

b) (i) \( 1-e^T + K(e^T - 1) < 0 \), (ii) \( 2-2K(e^T - 1) < 0 \), and (iii) \( 1+e^T + K(e^T - 1) < 0 \);

equivalently, either

a) (i) \( K > 1 \), (ii) \( K < \frac{1}{e^T - 1} \), and (iii) \( K > -\frac{e^T + 1}{e^T - 1} \);

or

b) (i) \( K < 1 \), (ii) \( K > \frac{1}{e^T - 1} \), and (iii) \( K < -\frac{e^T + 1}{e^T - 1} \).

Note that, since \( \frac{1}{e^T - 1} > 0 \) while \( -\frac{e^T + 1}{e^T - 1} < 0 \), for all \( T > 0 \), conditions (ii) and (iii) in part b are contradictory. On the other hand, since \( 1 > -\frac{e^T + 1}{e^T - 1} \), for all \( T > 0 \), condition (iii) in part a is automatically satisfied when condition (i) is satisfied. Therefore, for closed-loop stability we must have

\[ 1 < K < \frac{1}{e^T - 1} \]

which is the shaded region below. Note that there is no \( K \) which stabilize the system if \( T \geq \ln(2) \).
2. The transfer function of the system from the control input \( u \) to the output \( y \) is 
\[ G(s) = \frac{(s + 1)(s + 2)}{s(s - 1)}. \]
Since the system does not have any zeros at zero and/or at \( \pm j \), and does not have any unstable hidden modes (both modes, 0 and 1, of the system appear as poles of its transfer function), it is possible to design a controller which will achieve tracking of constant references and rejection of sinusoidal disturbances of angular frequency 1 rad/sec. If we take the first option (i.e., first design a continuous-time controller and then implement a discrete-time equivalent of it), the controller must have poles at \( \pm j \) (note that since the plant has a pole at zero, we do not need to include such a pole in the controller to track constant references), must stabilize the closed-loop system, and must be implemented in a unity feedback configuration as shown in Figure 1. With one open-loop pole at zero (which comes from the plant), the steady-state error in response to a ramp reference will not exceed (in absolute value) 10\% of the ramp slope provided that the closed-loop system is stable and the open-loop gain of the system is at least \( \frac{1}{0.1} = 10 \). Let us try a controller with transfer function 
\[ C(s) = \frac{n(s)}{s^2 + 1}, \]
which has the desired poles, where \( n(s) \) is a polynomial of degree at most 2 (so that the controller is proper).

If this is not sufficient, we can add more poles and increase the degree of \( n(s) \). A root-locus analysis (like the one in the solutions of Question 2 in Midterm Exam 2) will show that letting \( n(s) = K \) (i.e., a constant) is not sufficient to achieve closed-loop stability. By letting \( n(s) = K(s + 1) \), however, we obtain the same open-loop system as in the solutions of Question 2 in Midterm Exam 2. Therefore, as in there, by letting \( K = 5 \) we can satisfy both the closed-loop stability and the tracking (both for constant and ramp references) requirements. Therefore, our continuous-time controller transfer function is 
\[ C(s) = \frac{5(s + 1)}{s^2 + 1} \]
and the continuous-time implementation is as shown in Figure 1.

A minimal realization of the designed continuous-time controller is:
\[
\dot{z}(t) = A_c z(t) + B_c e(t)
\]
\[
u(t) = C_c z(t)
\]
where \( e(t) = r(t) - y(t) \), where \( r \) is the reference, \( z \) is the state of the continuous-time controller, 
\[ A_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C_c = \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \]
A discrete-time equivalent for this controller can be described as:
\[
z_d(k+1) = A_d z_d(k) + B_d e_d(k)
\]
\[
u_d(k) = C_c z_d(k)
\]
where \( e_d(k) = r(kT) - y(kT) \), \( z_d \) is the state of the discrete-time controller, \( u_d \) is the output of the controller which is applied to the input of the D/A whose output is applied to the control input of the plant,
\[ A_d = e^{A_c T} = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix} = \begin{bmatrix} 0.999 & 0.050 \\ -0.050 & 0.999 \end{bmatrix}, \]
and
\[ B_d = \int_0^T e^{A_c \tau} B_d \tau = \begin{bmatrix} 1 - \cos(T) \\ \sin(T) \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.050 \end{bmatrix}. \]
The following program implements this controller:

\[
\begin{bmatrix} 0.999 & 0.050 \\ -0.050 & 0.999 \end{bmatrix};
\]

\[
\begin{bmatrix} 0.001 \\ 0.050 \end{bmatrix};
\]

\[
\begin{bmatrix} 5 \\ 5 \end{bmatrix};
\]

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \text{ % initial condition}
\]

while 0==0 % enter an infinite loop
  
u = C \ast z
  
  write u
  
  read r, y
  
e = r - y
  
z = A \ast z + B \ast e
  
  wait % wait until the next sampling instant
end

Overall implementation of the actual controller is shown in Figure 2, where \( C_d(z) = C_c(zI - A_d)^{-1}B_d \).

If you take the alternate approach (i.e., directly design a discrete-time controller), you have to first determine the z-tfm of the equivalent discrete-time system from \( u_d \) to \( y_d \) in Figure 2, which is

\[
G_d(z) = \frac{z - 1}{z - \left( \frac{s^2 + 3s + 2}{s^2(s - 1)} \right)} = \frac{z^2 + (5e^T - 2T - 7)z + 6 + (2T - 5)e^T}{(z - 1)(z - e^T)} = \frac{(z - 0.9532)(z - 0.8904)}{(z - 1)(z - 1.0513)}.
\]

Then you have to design a controller tfm \( C_d(z) \), to be implemented as shown in Figure 2. To reject the disturbance \( d \), \( C_d(z) \) must have poles where \( d_d(k) = d(kT) \) have poles. Since \( Z(d_d(k)) = \frac{n_d(z)}{z^2 - 2z \cos(T) + 1} = \frac{n_d(z)}{z^2 - 1.9975z + 1} \) (here \( n_d(z) \) depends on the actual phase and magnitude of \( d(t) \), which are not known and are not needed), as a first try we can let \( C_d(z) = \frac{n_c(z)}{z^2 - 1.9975z + 1} \), where \( n_c(z) \) is a polynomial of degree at most 2 (more preferably of degree at most 1 to make the controller strictly proper). If this is not sufficient, more poles and zeros can be added (but, of course, you should not cancel any unstable poles or zeros of \( G_d(z) \)). Note that, since \( G_d(z) \) has a pole at 1, we do not need to add such a pole to \( C_d(z) \) (just like we didn’t need a pole at the origin for \( C(s) \) above) to track constant references. Here \( n_c(z) \) (and the additional poles of \( C_d(z) \), if needed) must be chosen such that the closed-loop discrete-time equivalent system (which can be described by the same block diagram as in Figure 1, with \( G(s), C(s), y, u, e, \) and \( r \) respectively replaced by \( G_d(z), C_d(z), y_d, u_d, e_d, \) and \( r_d \)) is internally stable and (in order to achieve the steady-state error requirement for ramp signals) \(|K_c^d| \geq 10\), where \( K_c^d := \lim_{z \to 1} (z - 1)G_d(z)C_d(z) \). Once such a \( C_d(z) \) is obtained, you should then obtain a realization (preferably a minimal one) of it and write a program to implement this realization.
3. The transfer function of the system from the control input $u$ to the output $y$ is $G(s) = \frac{s^2 + 1}{s(s - 1)}$. Since the system has zeros at $\pm j$, it is not possible to design a controller (continuous-time or discrete-time) to achieve rejection of sinusoidal disturbances of angular frequency 1 rad/sec while keeping the closed-loop system internally stable.